Non-Euclidean Contraction Theory for Network Systems

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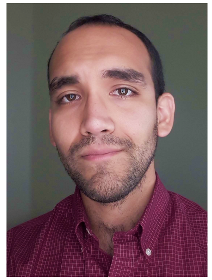

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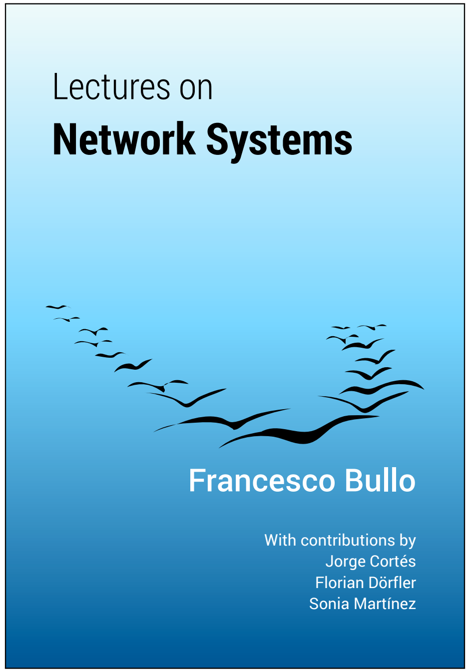

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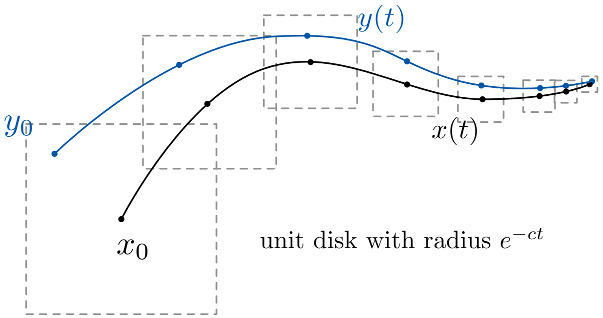
Prototypical nonlinear network systems

In chronological order

1. the Lotka-Volterra population dynamics (Lotka, 1920; Volterra, 1928),
2. Matrosov-Bellman interconnection of stable systems (and the method of vector Lyapunov functions) (Bellman, 1962; Matrosov, 1962),
3. Kuramoto oscillators (Kuramoto, 1975) and coupled swing equations (Bergen and Hill, 1981),
4. Yorke network epidemic model (Lajmanovich and Yorke, 1976),
5. Hopfield and cellular neural networks (Hopfield, 1982),
6. Daganzo cell transmission model for traffic networks (Daganzo, 1994),
7. Chua's diffusively-coupled dynamical systems (Wu and Chua, 1995), and
8. compartmental systems in biology, medicine, and ecology (Sandberg, 1978; Maeda et al., 1978), earlier models fading back into the past.

Contraction theory: a brief overview
Definition

f is contractive if its flow is a contraction map



Contraction theory: a brief overview
Historical notes

- **Origins**

  G. Dahlquist. *Stability and error bounds in the numerical integration of ordinary differential equations*. PhD thesis, (Reprinted in Trans. Royal Inst. of Technology, No. 130, Stockholm, Sweden, 1959), 1958

  B. P. Demidovič. Dissipativity of a nonlinear system of differential equations. *Uspekhi Matematicheskikh Nauk*, 16(3(99)):216, 1961

  C. A. Desoer and H. Haneda. The measure of a matrix as a tool to analyze computer algorithms for circuit analysis. *IEEE Transactions on Circuit Theory*, 19(5):480–486, 1972.

- **Application in control theory**: W. Lohmiller and J.-J. E. Slotine. On contraction analysis for non-linear systems. *Automatica*, 34(6):683–696, 1998.

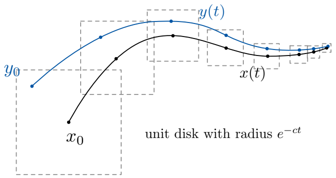
- **Non-Euclidean contraction**: S. Coogan. A contractive approach to separable Lyapunov functions for monotone systems. *Automatica*, 106:349–357, 2019.

- **Reviews**:

  M. Di Bernardo, D. Fiore, G. Russo, and F. Scafuti. Convergence, consensus and synchronization of complex networks via contraction theory. In J. Lü, X. Yu, G. Chen, and W. Yu, editors, *Complex Systems and Networks*, pages 313–339. Springer, 2016. ISBN 978-3-662-47824-0.

  Z. Aminzare and E. D. Sontag. Contraction methods for nonlinear systems: A brief introduction and some open problems. In *IEEE Conf. on Decision and Control*, pages 3835–3847, Dec. 2014.

Contraction theory: a brief overview
Properties of contracting systems



Highly ordered **transient** and **asymptotic** behavior:

1. initial conditions are forgotten, and
   monotonic decrease (no overshoot) in distance between trajectories
2. time-invariant f: unique globally exponential stable equilibrium
   two natural Lyapunov functions
3. periodic f: unique globally exponentially stable periodic, and solution
   contracting system entrain to periodic inputs
4. robustness properties:
   input-to-state stability
   finite input-state gain
   contraction margin wrt unmodeled dynamics
   input-to-state stability under delayed dynamics
   $\Longrightarrow$ contractivity rate is natural measure/indicator of robust stability
5. accurate numerical integration and fixed point computation

Contraction theory: a brief overview
Contraction theory vs Lyapunov stability theory

Contraction theory combines in unified coherent framework results from:

1. stability notions and Lyapunov functions for incremental stability or stability of equilibrium points and trajectories,
2. Banach contraction and Brower fixed point theorems,
3. monotone systems theory, and
4. geometry of Banach, Riemannian and Finsler spaces

nonlinear robust stability theory

Outline

Contraction analysis for scalar vector field

$f : \mathbb{R} \to \mathbb{R}$ is **one-sided Lipschitz (osL) continuous** if there exists $b \in \mathbb{R}$ s.t.

$$\begin{aligned} & f'(x) \le b, && \forall x && (\textbf{d-osL}) \\ \iff\quad & f(x) - f(y) \le b(x-y), && \forall x > y && \\ \iff\quad & (x-y)(f(x)-f(y)) \le b(x-y)^2, && \forall x, y && (\textbf{osL}) \end{aligned}$$

- f is osL with $b = 0$ iff f weakly decreasing
- if f is Lipschitz with bound ℓ, then f is osL with $b = \ell$
- For

$$\dot{x} = f(x) \tag{1}$$

the Grönwall lemma implies $|x(t) - y(t)| \le e^{bt}|x(0) - y(0)|$

Contraction theorem on $\mathbb{R}^n$ and ℓ_2 norm

For $x \in \mathbb{R}^n$ and differentiable time-dep

$$\dot{x} = f(t, x) \tag{2}$$

For $P = P^\top \succ 0$, define $\|x\|_P^2 = x^\top P x$

Main equivalences:

1. **osL** : $(f(t,x) - f(t,y))^\top P(x-y) \le b\|x-y\|_P^2$, for all x, y, t
2. **d-osL** : $PDf(t,x) + Df(t,x)^\top P \preceq 2bP$ for all x, t
3. **d-IS** : $D^+\|x(t) - y(t)\|_P \le b\|x(t) - y(t)\|_P$, for all soltns $x(\cdot), y(\cdot)$
4. **IS** : $\|x(t) - y(t)\|_P \le e^{b(t-t_0)}\|x(t_0) - y(t_0)\|_P$, for all soltns $x(\cdot), y(\cdot)$

If f not differentiable, then **osL** $\iff$ **d-IS** $\iff$ **IS**

Equivalent rewriting with inner products

For $x \in \mathbb{R}^n$ and differentiable time-dep

$$\dot{x} = f(t, x) \tag{3}$$

For $P = P^\top \succ 0$, define $\langle\!\langle x, y \rangle\!\rangle_P = x^\top P y$ and $\|x\|_P^2 = x^\top P x$,

Main equivalences:

1. **osL** : $\langle\!\langle f(t,x) - f(t,y), x - y \rangle\!\rangle_P \le b\|x-y\|_P^2$, for all $x, y, t \ge 0$
2. **d-osL** : $\langle\!\langle Df(t,x)v, v \rangle\!\rangle_P \le b\|v\|_P^2$ for all $x, v, t \ge 0$,
3. **d-IS** : $D^+\|x(t) - y(t)\|_P \le b\|x(t) - y(t)\|_P$, for all soltns $x(\cdot), y(\cdot)$
4. **IS** : $\|x(t) - y(t)\|_P \le e^{b(t-t_0)}\|x(t_0) - y(t_0)\|_P$, for all soltns $x(\cdot), y(\cdot)$

Linear algebra detour: Matrix measures

The **matrix measure** of $A \in \mathbb{R}^{n\times n}$ wrt to $\|\cdot\|$:

$$\mu(A) := \lim_{h\to 0^+} \frac{\|I_n + hA\| - 1}{h}$$

$$\mu_2(A) = \tfrac{1}{2}\lambda_{\max}(A + A^\top)$$

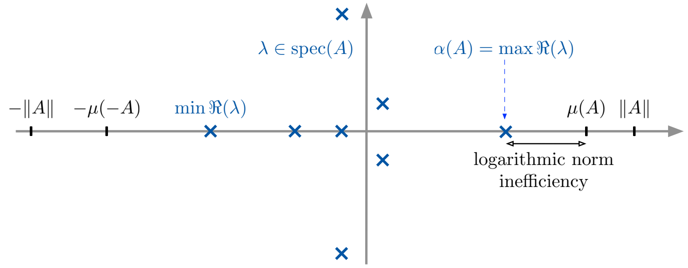
$$\mu_1(A) = \max_j \left(a_{jj} + \sum\nolimits_{i\neq j} |a_{ij}|\right) \qquad \mu_\infty(A) = \max_i \left(a_{ii} + \sum\nolimits_{j\neq i} |a_{ij}|\right)$$

Basic properties:

subadditivity:	$\mu(A+B) \leq \mu(A) + \mu(B)$	
scaling:	$\mu(bA) = b\mu(bA),$	$\forall b \geq 0$
convexity:	$\mu(\theta A + (1-\theta)B) \leq \theta\mu(A) + (1-\theta)\mu(B),$	$\forall \theta \in [0,1]$

T. Ström. On logarithmic norms. *SIAM Journal on Numerical Analysis*, 12(5):741–753, 1975.

norm/spectrum: $\quad \mathrm{Re}(\lambda) \leq \mu(A) \leq \|A\|, \qquad \forall \lambda \in \mathrm{spec}(A)$



A norm $\|\cdot\|$ is

1. **logarithmically optimal** for A if $\mu(A) = \alpha(A)$, and
2. **logarithmically ϵ-efficient** for A if $\alpha(A) \leq \mu(A) \leq \alpha(A) + \epsilon$.

ℓ_2 :	$\forall A,$	$\exists P \succ 0,$ s.t. $\|\cdot\|_P$ is efficient
$\ell_p, p \in [1,\infty]$:	$\forall M$ Metzler,	$\exists \eta > \mathbb{0}_n,$ s.t. $\|\cdot\|_{p,[\eta]}$ is efficient

Connection between osL and matrix measure

The best osL constant:

$$\begin{aligned}\mathrm{osL}_2(f) &:= \sup_{x\neq y} \frac{\langle\!\langle f(x) - f(y), x - y\rangle\!\rangle}{\|x-y\|_2^2} \\ &= \sup_x \mu_2(Df(x)) \qquad \text{if } f \text{ differentiable}\end{aligned}$$

Proof for affine vector fields:

$$\begin{aligned}\mathrm{osL}_2(Ax+b) &= \sup_{x\neq y} \frac{\langle\!\langle Ax - Ay, x-y\rangle\!\rangle}{\|x-y\|_2^2} \\ &= \sup_{x\neq \mathbb{0}_n} \frac{x^\top A x}{x^\top x} && \text{(Rayleigh quotient)} \\ &= \tfrac{1}{2}\lambda_{\max}(A + A^\top) && \text{(sup of numerical range)} \\ &= \mu_2(A)\end{aligned}$$

Weak pairing

A **weak pairing** (WP) is $[\![\cdot,\cdot]\!] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying

1. $[\![x_1 + x_2, y]\!] \leq [\![x_1, y]\!] + [\![x_2, y]\!]$ and $x \mapsto [\![x, y]\!]$ is continuous,
2. $[\![bx, y]\!] = [\![x, by]\!] = b[\![x, y]\!]$ for $b \geq 0$ and $[\![-x, -y]\!] = [\![x, y]\!]$,
3. $[\![x, x]\!] > 0$, for all $x \neq \mathbb{0}_n$,
4. $|\,[\![x, y]\!]\,| \leq [\![x, x]\!]^{1/2}\,[\![y, y]\!]^{1/2}$,

Given norm $\|\cdot\|$, compatibility: $[\![x, x]\!] = \|x\|^2$ for all x

Sup of non-Euclidean numerical range: $\qquad \mu(A) = \sup_{x\neq 0} \dfrac{[\![Ax, x]\!]}{\|x\|^2}$

Norm derivative formula: $\qquad \tfrac{1}{2}D^+\|x(t)\|^2 = [\![\dot{x}(t), x(t)]\!]$

G. Lumer. Semi-inner-product spaces. *Transactions of the American Mathematical Society*, 100: 29–43, 1961.

Norm	WP	Matrix measure
$\|x\|_{2,P^{1/2}} = \sqrt{x^\top P x}$	$[\![x,y]\!]_{2,P^{1/2}} = x^\top P y$	$\mu_{2,P^{1/2}}(A) = \min\{b \mid A^\top P + PA \preceq 2bP\} = \frac{1}{2}\lambda_{\max}(PAP^{-1}+A^\top) = \max_{\|x\|_P=1} x^\top PAx$
$\|x\|_p = \Big(\sum_i \lvert x_i\rvert^p\Big)^{1/p}$, $p \in]1,\infty[$	$[\![x,y]\!]_p = \|y\|_p^{2-p}(y \circ \lvert y\rvert^{p-2})^\top x$	$\mu_p(A) = \max_{\|x\|_p=1} (x \circ \lvert x\rvert^{p-2})^\top Ax$
$\|x\|_1 = \sum_i \lvert x_i\rvert$	$[\![x,y]\!]_1 = \|y\|_1 \operatorname{sign}(y)^\top x$	$\mu_1(A) = \max_{j\in\{1,\dots,n\}} \Big(a_{jj} + \sum_{i\neq j} \lvert a_{ij}\rvert\Big) = \sup_{\|x\|_1=1} \operatorname{sign}(x)^\top Ax$
$\|x\|_\infty = \max_i \lvert x_i\rvert$	$[\![x,y]\!]_\infty = \max_{i\in I_\infty(y)} y_i x_i$	$\mu_\infty(A) = \max_{i\in\{1,\dots,n\}} \Big(a_{ii} + \sum_{j\neq i} \lvert a_{ij}\rvert\Big) = \max_{\|x\|_\infty=1} \max_{i\in I_\infty(x)} x_i(Ax)_i$

Table of norms, WPs, and matrix measures for weighted ℓ_2, ℓ_p for $p \in]1,\infty[$, ℓ_1, and ℓ_∞ norms. Note: $I_\infty(x) = \{i \in \{1,\dots,n\} \mid |x_i| = \|x\|_\infty\}$.

Contraction theorem on $\mathbb{R}^n$ with arbitrary norm

For $x \in \mathbb{R}^n$ and differentiable time-dep

$$\dot{x} = f(t,x) \tag{4}$$

For norm $\|\cdot\|$ with matrix measure $\mu(\cdot)$ and compatible WP $[\![\cdot,\cdot]\!]$,

Main equivalences:

1. **osL** : $[\![f(t,x) - f(t,y), x-y]\!] \leq b\|x-y\|^2$ for all x, y, $t \geq 0$,
2. **d-osL** : $[\![Df(t,x)v, v]\!] \leq b\|v\|^2$, for all v, x, $t \geq 0$, or $\mu(Df(t,x)) \leq b$, for all $x, t \geq 0$,
3. **d-IS** : $D^+\|x(t) - y(t)\| \leq b\|x(t) - y(t)\|$, for soltns $x(\cdot), y(\cdot)$,
4. **IS** : $\|x(t) - y(t)\| \leq e^{b(t-t_0)}\|x(t_0) - y(t_0)\|$, for all soltns $x(\cdot), y(\cdot)$

Measure bound	Demidovich condition	One-sided Lipschitz condition
$\mu_{2,P}(Df(x)) \leq b$	$PDf(x) + Df(x)^\top P \preceq 2bP$	$(x-y)^\top P(f(x) - f(y)) \leq b\|x-y\|_{P^{1/2}}^2$
$\mu_p(Df(x)) \leq b$	$(v \circ \lvert v\rvert^{p-2})^\top Df(x)v \leq b\|v\|_p^p$	$((x-y)\circ\lvert x-y\rvert^{p-2})^\top(f(x) - f(y)) \leq b\|x-y\|_p^p$
$\mu_1(Df(x)) \leq b$	$\operatorname{sign}(v)^\top Df(x)v \leq b\|v\|_1$	$\operatorname{sign}(x-y)^\top(f(x) - f(y)) \leq b\|x-y\|_1$
$\mu_\infty(Df(x)) \leq b$	$\max_{i\in I_\infty(v)} v_i(Df(x)v)_i \leq b\|v\|_\infty^2$	$\max_{i\in I_\infty(x-y)} (x_i - y_i)(f_i(x) - f_i(y)) \leq b\|x-y\|_\infty^2$

Table of equivalences between measure bounded Jacobians, differential Demidovich and one-sided Lipschitz conditions. Note: $I_\infty(v) = \{i \in \{1,\dots,n\} \mid |v_i| = \|v\|_\infty\}$.

J. A. Jacquez and C. P. Simon. Qualitative theory of compartmental systems. *SIAM Review*, 35(1):43–79, 1993.

H. Qiao, J. Peng, and Z.-B. Xu. Nonlinear measures: A new approach to exponential stability analysis for Hopfield-type neural networks. *IEEE Transactions on Neural Networks*, 12(2):360–370, 2001.

G. Como, E. Lovisari, and K. Savla. Throughput optimality and overload behavior of dynamical flow networks under monotone distributed routing. *IEEE Transactions on Control of Network Systems*, 2(1):57–67, 2015.

Metzler matrices and monotone systems

- For Metzler M and monotonic $\|\cdot\|$, $\mu(M) = \sup_{x \geq \mathbb{0}_n} \dfrac{[\![Ax, x]\!]}{\|x\|}$.
- For $\eta, \xi \in \mathbb{R}^n_{>0}$,

$$\mu_{1,[\eta]}(M) = \max(\eta^\top M[\eta]^{-1}) = \min\{b \in \mathbb{R} \mid \eta^\top M \leq b\eta^\top\}$$
$$\mu_{\infty,[\xi]^{-1}}(M) = \max([\xi]^{-1}M\xi) = \min\{b \in \mathbb{R} \mid M\xi \leq b\xi\}$$

f **monotone** if $Df(x)$ Metzler for all x

1. **osL** : $[\![f(x) - f(y), x-y]\!] \leq b\|x-y\|^2$ for all $x \geq y$
2. **d-osL** : $[\![Df(x)v, v]\!] \leq b\|v\|^2$, for all $v \geq 0$ and x

$\mu_{1,[\eta]}(Df(x)) \leq b$	$\eta^\top Df(x) \leq b\eta^\top$	$\eta^\top(f(x) - f(y)) \leq b\eta^\top(x-y)$ for all $x \geq y$
$\mu_{\infty,[\xi]^{-1}}(Df(x)) \leq b$	$Df(x)\xi \leq b\xi$	$f(x) - f(y) \leq b(x-y)$ for all $x = y + \beta\xi, \beta > 0$

Infinitesimally contracting systems

Given vector field f over normed space with WP:

$$[\![f(t,x) - f(t,y), x - y]\!] \leq b\|x-y\|^2, \qquad \text{for all } x, y, t \geq 0$$

$b = -c$, $c > 0$	:	strongly contracting with rate c
$b = 0$ and strict inequality	:	strictly contracting
$b = 0$	:	weakly contracting (or non-expansive)

For differentiable V over convex set C, equivalent statements:

1. V is **strongly convex** with parameter m
2. $-\operatorname{grad} V$ is **strongly contracting** with rate m wrt ℓ_2, that is

$$\big(-\operatorname{grad} V(x) + \operatorname{grad} V(y)\big)^\top (x-y) \leq -m\|x-y\|_2^2$$

Outline

1. Main equivalence theorem
2. Analysis of dynamic behavior and robustness
3. Network contraction theory: Small gain theorems
4. Network contraction theory: Weakly contracting systems
5. Network contraction theory: Semi-contracting systems

Globally exponentially stable equilibrium

For time-invariant vector field f and norm $\|\cdot\|$

1. there exists a convex and f-invariant set C,
2. f is strongly contracting with rate c on C

Then

1. flow of f is a contraction, i.e.,
   distance between solutions exponentially decreases with rate c
2. there exists an equilibrium x^*, unique, globally exponentially stable
   with global Lyapunov functions

$$x \mapsto \|x - x^*\|^2 \quad \text{and} \quad x \mapsto \|f(x)\|^2$$

Globally exponentially stable periodic orbits

a.k.a., Entrainment to periodic inputs

For time-varying vector field f and norm $\|\cdot\|$

1. there exists a convex, closed f-invariant set C,
2. f is strongly contracting with rate c on C,
3. f is T-periodic.

Then

1. there exists a unique periodic solution $x^* : \mathbb{R}_{\geq 0} \to C$ with period T
2. for every initial condition $x_0 \in C$,

$$\|x(t, x_0) - x^*(t)\| \leq \mathrm{e}^{-ct}\|x_0 - x^*(0)\| \qquad (5)$$

G. Russo, M. Di Bernardo, and E. D. Sontag. Global entrainment of transcriptional systems to periodic inputs. *PLoS Computational Biology*, 6(4):e1000739, 2010.

Input-state stability and gain of contracting systems

For a time and input-dependent vector f,

$$\dot{x} = f(t, x, u(t)), \qquad x(0) = x_0 \in \mathbb{R}^n, u(t) \in \mathbb{R}^k \tag{6}$$

Assume $\|\cdot\|_{\mathcal{X}}$ with compatible $[\![\cdot,\cdot]\!]_{\mathcal{X}}$, a norm $\|\cdot\|_{\mathcal{U}}$, and $c, \ell > 0$ such that

- **osL**: $[\![f(t,x,u) - f(t,y,u), x-y]\!]_{\mathcal{X}} \leq -c\|x-y\|_{\mathcal{X}}^2$, for all t, x, y, u,
- **Lip**: $\|f(t,x,u) - f(t,x,v)\|_{\mathcal{X}} \leq \ell\|u-v\|_{\mathcal{U}}$, for all t, x, u, v.

Input-state stability and gain of contracting systems

Then

1. any two soltns $x(t)$ and $y(t)$ to (6) with inputs u_x, u_y

$$D^+\|x(t) - y(t)\|_{\mathcal{X}} \leq -c\|x(t) - y(t)\|_{\mathcal{X}} + \ell\|u_x(t) - u_y(t)\|_{\mathcal{U}}$$

2. f is **incrementally input-to-state stable**, i.e., for all x_0, y_0

$$\|x(t) - y(t)\|_{\mathcal{X}} \leq e^{-ct}\|x_0 - y_0\|_{\mathcal{X}} + \frac{\ell(1-e^{-ct})}{c} \sup_{\tau\in[0,t]} \|u_x(\tau) - u_y(\tau)\|_{\mathcal{U}}$$

3. f has **incremental $\mathcal{L}^q_{\mathcal{X},\mathcal{U}}$ gain equal to ℓ/c, for $q \in [1,\infty]$**,

$$\|x(\cdot) - y(\cdot)\|_{\mathcal{X},q} \leq \frac{\ell}{c}\|u_x(\cdot) - u_y(\cdot)\|_{\mathcal{U},q} \qquad (\text{for } x_0 = y_0)$$

Signal norms and system gains

Given norm $\|\cdot\|_{\mathcal{X}}$ on $\mathbb{R}^n$ (or $\|\cdot\|_{\mathcal{U}}$ on $\mathbb{R}^k$),

- $\mathcal{L}^q_{\mathcal{X}}$, $q \in [1,\infty]$, is vector space of continuous signals, $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, with well-defined bounded norm

$$\|x(\cdot)\|_{\mathcal{X},q} = \begin{cases} \left(\int_0^\infty \|x(t)\|_{\mathcal{X}}^q dt\right)^{1/q} & \text{if } q \in [1,\infty[\\ \sup_{t\geq 0}\|x(t)\|_{\mathcal{X}} & \text{if } q = \infty \end{cases} \tag{7}$$

- Input-state system has **$\mathcal{L}^q_{\mathcal{X},\mathcal{U}}$-induced gain** upper bounded by $\gamma > 0$ if, for all $u \in \mathcal{L}^q_{\mathcal{U}}$, the state x from zero initial state satisfies

$$\|x(\cdot)\|_{\mathcal{X},q} \leq \gamma\,\|u(\cdot)\|_{\mathcal{U},q} \tag{8}$$

Robustness to unmodeled dynamics

Given a norm $\|\cdot\|$, consider

$$\dot{x} = f(t,x) + g(t,x) \tag{9}$$

If f has one-sided Lipschitz constant $-c < 0$ and
g has one-sided Lipschitz constant $d > 0$, then

1. **(contractivity under perturbations)** if $d < c$, then $f + g$ is strongly contracting with rate $c - d$,
2. **(equilibrium point under perturbations)** if additionally f and g are time-invariant, then the unique equilibrium point x^* of f and x^{**} of $f+g$ satisfy

$$\|x^* - x^{**}\| \leq \frac{\|g(x^*)\|}{c-d} \tag{10}$$

Outline

Networks of contracting systems

Interconnected subsystems

$$\dot{x}_i = f_i(t, x_i, x_{-i}), \qquad \text{for } i \in \{1, \dots, n\}, \tag{11}$$

where $x_i \in \mathbb{R}^{N_i}$, $N = \sum_{i=1}^n N_i$, and $x_{-i} \in \mathbb{R}^{N-N_i}$.

If

- **osL**: $[\![f_i(t, x_i, x_{-i}) - f_i(t, y_i, x_{-i}), x_i - y_i]\!]_i \leq -c_i \|x_i - y_i\|_i^2$
- **Lip**: $\|f_i(t, x_i, x_{-i}) - f_i(t, x_i, y_{-i})\|_i \leq \sum_{j=1, j\neq i}^n \gamma_{ij} \|x_j - y_j\|_j$
- the gain matrix $\begin{bmatrix} -c_1 & \dots & \gamma_{1n} \\ \vdots & & \vdots \\ \gamma_{n1} & \dots & -c_n \end{bmatrix}$ is Hurwitz

then the **interconnected system** is strongly contracting
wrt appropriate composite norm and with rate = (-) abscissa gain matrix

Hurwitz Metzler Theorem

1. M is Hurwitz,
2. there exists $\eta \in \mathbb{R}^n_{>0}$ such that $\eta^\top M < \mathbb{0}_n^\top$ or, equivalently, $\mu_{1,[\eta]}(M) < 0$,
3. there exists $\xi \in \mathbb{R}^n_{>0}$ such that $M\xi < \mathbb{0}_n$ or, equivalently, $\mu_{\infty,[\xi]^{-1}}(M) < 0$, and
4. there exists a diagonal $P = P^\top \succ 0$ satisfying $M^\top P + PM \prec 0$ or, equivalently, $\mu_{2,P^{1/2}}(M) < 0$.

1. DAG interconnections of contracting systems are strongly contracting
2. 2-dimensional matrix: small gain condition $c_1 c_2 > \ell_1 \ell_2$

X. Duan, S. Jafarpour, and F. Bullo. Graph-theoretic stability conditions for Metzler matrices and monotone systems. *SIAM Journal on Control and Optimization*, 59(5):3447–3471, 2021.

Networks of ISS systems

Interconnections scalar ISS subsystems

$$\dot{x}_i = -a_i(x_i) + \sum_{j\neq i} \gamma_{ij}(x_j) + u_i, \qquad \text{for } i \in \{1, \dots, n\}. \tag{12}$$

where a_i are of class $\mathcal{K}_\infty$ and γ_{ij} are of class $\mathcal{K}$. Define

$$A_i(x) = a_i(x_i), \qquad \text{and } \Gamma_i(x) = \sum_{j\neq i} \gamma_{ij}(x_j)$$

If there exist $\eta \in \mathbb{R}^n_{>0}$ and $c > 0$ satisfying

$$\eta^\top \big(A(v) - A(w)\big) \geq \eta^\top \big(\Gamma(v) - \Gamma(w) + c(v - w)\big), \qquad \text{for all } v \geq w \geq \mathbb{0}_n$$

then the **interconnected system** is strongly contracting
with respect to $\|\cdot\|_{1,[\eta]}$ and with rate c

Proof: $\text{osL}_{1,[\eta]}(f) \leq b$ if and only if $\eta^\top \big(f(x) - f(y)\big) \leq b\eta^\top (x - y)$

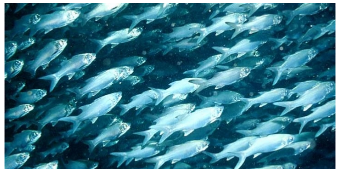
Outline

1. Main equivalence theorem
2. Analysis of dynamic behavior and robustness
3. Network contraction theory: Small gain theorems
4. Network contraction theory: Weakly contracting systems
5. Network contraction theory: Semi-contracting systems

Contraction theory for networks

Challenge: many real-world networks are not contracting.





conservation law: $\mathbb{1}_n^\top x(t) = \text{const}$ **invariance, symmetry:** $f(x + \alpha \mathbb{1}_n) = f(x)$

For a vector field f and positive vectors $\eta, \xi \in \mathbb{R}^n_{>0}$,

conservation law	$\eta^\top f(x) = \eta^\top f(y)\ \forall x, y$	$\iff$	$\eta^\top Df(x) = 0\ \forall x$
translation invariance	$f(x + \alpha\xi) = f(x)\ \forall x, \alpha$	$\iff$	$Df(x)\xi = 0\ \forall x$

If f satisfyies a conservation or resp. invariance, then

1. $\text{osL}(f) \geq 0$,
2. if, additionally, f is monotone, then $\text{osL}_{1,[\eta]}(f) = 0$ or resp. $\text{osL}_{\infty,[\xi]^{-1}}(f) = 0$

Weakly-contracting systems

Definition and examples

$\dot{x} = f(t, x)$ is **weakly-contracting** wrt $\|\cdot\|$:

$$\text{osL}(f) \leq 0$$

1. Lotka-Volterra population dynamics (Lotka, 1920; Volterra, 1928) (**ℓ_1-norm for mutualistic**)
2. Kuramoto oscillators (Kuramoto, 1975) and coupled swing equations (Bergen and Hill, 1981) (**ℓ_1-norm and ℓ_∞-norm**)
3. Daganzo's cell transmission model for traffic networks (Daganzo, 1994), (**ℓ_1-norm for non-FIFO intersection**)
4. compartmental systems in biology, medicine, and ecology (Sandberg, 1978; Maeda et al., 1978). (**ℓ_1-norm**)
5. saddle-point dynamics for optimization of weakly-convex functions (Arrow et al., 1958). (**ℓ_2-norm**)

Weakly-contracting systems

Part I: Dichotomy in asymptotic behavior

Theorem: Dichotomy for weakly-contracting systems For a weakly-contracting system $\dot{x} = f(x)$, either

1. f has no equilibrium and every trajectory is unbounded, or
2. f has at least one equilibrium x^* and every trajectory is bounded.

Weakly-contracting systems

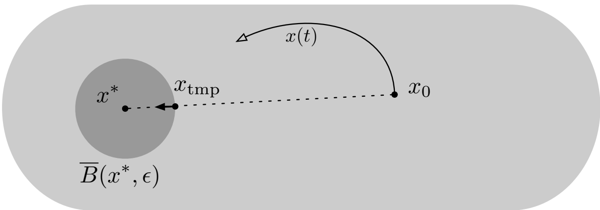
Part II: bounded trajectory case

Theorem If $\dot{x} = f(x)$ is weakly-contracting and f has at least one equilibrium x^* then:

(i) each equilibrium x^{**} is stable with weak Lyapunov function $x \mapsto \|x - x^{**}\|$,

(ii) if the norm $\|\cdot\|$ is a (p, R)-norm, $p \in \{1, \infty\}$ and f is piecewise real analytic, then every trajectory converges to the set of equilibria,

(iii) x^* is locally asy stable $\implies$ x^* is globally asy stable.



Example: Distributed primal-dual algorithm

Optimization problem

$$\min_{x \in \mathbb{R}^k} f(x) = \min_{x \in \mathbb{R}^k} \sum_{i=1}^n f_i(x)$$

Distributed implementation

- n agents communicate over a undirected weighted graph G,
- agent i have access to function f_i and can exchange x_i with its neighbors.

$$\min_{x \in \mathbb{R}^k} \sum_{i=1}^n f_i(x_i)$$
$$x_1 = x_2 = \ldots = x_n$$

In matrix form by assuming $x = (x_1^\top, \ldots, x_n^\top)^\top \in \mathbb{R}^{nk}$:

$$\min_{x \in \mathbb{R}^k} \sum_{i=1}^n f_i(x_i)$$
$$(L \otimes I_k)x = \mathbb{0}_{nN}$$

Example: Distributed primal-dual algorithm

If each f_i is continuously differentiable in x_i:

Lagrangian
$\mathcal{L}(x, \nu) = \sum_{i=1}^n f_i(x_i) + \nu^\top (L \otimes I_k)x$

Distributed primal-dual algorithm (component form):

$$\dot{x}_i = -\frac{\partial \mathcal{L}}{\partial x_i} = -\nabla f_i(x_i) - \sum_{j=1}^n a_{ij}(\nu_i - \nu_j),$$
$$\dot{\nu}_i = \frac{\partial \mathcal{L}}{\partial \nu_i} = \sum_{j=1}^n a_{ij}(x_i - x_j)$$

Distributed primal-dual algorithm (vector form):

$$\dot{x} = -\nabla f(x) - (L \otimes I_k)\nu,$$
$$\dot{\nu} = (L \otimes I_k)x$$

Example: Distributed primal-dual algorithm

1. f has a minimum $x^* \in \mathbb{R}^k$,
2. f_i is twice differentiable, $\nabla^2 f_i(x) \succeq 0$ for all x, and $\nabla^2 f_i(x^*) \succ 0$, and
3. the undirected weighted graph G is connected with Laplacian L.

Theorem: Distributed primal-dual dynamics The distributed primal-dual algorithm

1. is weakly-contracting wrt ℓ_2-norm,
2. $(x(t), \nu(t)) \to (\mathbb{1}_n \otimes x^*, \mathbb{1}_n \otimes \nu^*)$, with $\nu^* = \sum_{i=1}^n \nu_i(0)$,
3. exponential convergence rate is $-\alpha_{\mathrm{ess}}\left(\begin{bmatrix} -\nabla^2 f(x^*) & -L \otimes I_k \\ L \otimes I_k & \mathbb{0} \end{bmatrix}\right)$ where

$$\alpha_{\mathrm{ess}}(A) := \max\{\Re(\lambda) \mid \lambda \in \mathrm{spec}(A) \setminus \{0\}\}.$$

Proof: $\mu_2(A) = 0$ and there exists locally asy stable equilibrium point

Outline

Semi-contracting systems

Semi-norms

Definition: Semi-norms

$|||\cdot|||$ is a **semi-norm** if

1. $|||cv||| = |c|\,|||v|||$, for every $v \in \mathbb{R}^n$ and $c \in \mathbb{R}$;
2. $|||v+w||| \leq |||v||| + |||w|||$, for every $v, w \in \mathbb{R}^n$.

- define the subspace $\operatorname{Ker} |||\cdot||| = \{v \in \mathbb{R}^n \mid |||v||| = 0\}$
- Example: for $k < n$, $R \in \mathbb{R}^{k \times n}$, and norm $\|\cdot\|$, we get $|||x|||_R = \|Rx\|$

Semi-contracting systems

Matrix semi-measures

The **matrix semi-measure** of $A \in \mathbb{R}^{n\times n}$ wrt $|||\cdot|||$:

$$\mu_{|||\cdot|||}(A) = \lim_{h \to 0^+} \frac{|||I_n + hA||| - 1}{h}.$$

if $\operatorname{Ker} |||\cdot|||$ is invariant under A,

then $\Re(\lambda) \leq \mu_{|||\cdot|||}(A)$, for every $\lambda \in \operatorname{spec}_{\operatorname{Ker} |||\cdot|||^{\perp}}(A^\top)$.

Semi-norms for network systems

For undirected G with edge set $\mathcal{E}$, incidence matrix B and Laplacian L

$$|||x|||_{\mathcal{E}} := \max_{(i,j)\in\mathcal{E}} |x_i - x_j| = \|B^\top x\|_\infty$$

For connected graphs $\operatorname{Ker} |||\cdot|||_{\mathcal{E}} = \operatorname{span}\{\mathbb{1}_n\}$

The orthogonal projection $\Pi_n : \mathbb{R}^n \to \operatorname{span}\{\mathbb{1}_n\}^\perp$

$$\Pi_n = I_n - \tfrac{1}{n}\mathbb{1}_n\mathbb{1}_n^\top = \begin{bmatrix} \frac{n-1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & \frac{n-1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & \frac{n-1}{n} \end{bmatrix} \succeq 0$$

Logarithmically-optimal semi-norm

$$\mu_{2,\Pi_n}(-L) = -\lambda_2(L)$$

Semi-contracting systems

Definition and examples

$\dot{x} = f(t,x)$ is **semi-contracting** wrt the semi-norm $|||\cdot|||$ with rate $c > 0$:

$$\mathrm{osL}_{|||\cdot|||}(f) \leq -c$$

or, for differentiable systems, $\mu_{|||\cdot|||}(Df(t,x)) \leq -c$

1. Kuramoto oscillators (Kuramoto, 1975) and coupled swing equations (Bergen and Hill, 1981), (**ℓ_1-norm**)
2. Chua's diffusively-coupled circuits (Wu and Chua, 1995), (**ℓ_2-norm**)
3. morphogenesis in developmental biology (Turing, 1952), (**ℓ_1-norm, over some param ranges**)
4. Goodwin model for oscillating auto-regulated gene (Goodwin, 1965). (**ℓ_1-norm, over some param ranges**)

Semi-contracting systems

Semi-contraction and asymptotic behavior

Consider $\dot{x} = f(t,x)$ with f continuously differentiable in x and assume

- f is semi-contracting wrt the semi-norm $|||\cdot|||$ with rate $c > 0$, and
- **(Affine invariance)**: there exists x^* such that $f(t, x^* + \mathrm{Ker}\,|||\cdot|||) \subseteq \mathrm{Ker}\,|||\cdot|||$

Then,

1. for every trajectory $x(t)$,

$$|||x(t) - x^*||| \leq e^{-ct}|||x(0) - x^*|||, \qquad \text{for every } t \geq 0.$$

2. every trajectory converges to $x^* + \mathrm{Ker}\,|||\cdot|||$.

Example: Diffusively-coupled oscillators

- n agents connected by a weighted undirected graph G,
- identical internal dynamics $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^k \to \mathbb{R}^k$

$$\dot{x}_i = f(t, x_i) - \textstyle\sum_{j=1}^n a_{ij}(x_i - x_j), \quad i \in \{1, \dots, n\}$$

- **synchronization**:

$$\lim_{t\to\infty} \|x_i - x_j\| = 0 \qquad \text{for every } i, j$$

- synchronization of diffusively-coupled oscillators:
  1. contractivity of the internal dynamics
  2. strength of the diffusive coupling

Introduce **local-global mixed norm:** $(2,p)$-tensor norm on $\mathbb{R}^{nk} = \mathbb{R}^n \otimes \mathbb{R}^k$

$$\|u\|_{(2,p)} = \inf\Big\{ \Big(\textstyle\sum_{i=1}^r \|v^i\|_2^2 \|w^i\|_p^2\Big)^{\frac{1}{2}} \;\Big|\; u = \textstyle\sum_{i=1}^r v^i \otimes w^i \Big\}.$$

- **global norm**: ℓ_2-norm for the interactions between agents
- **local norm**: ℓ_p-norm for internal dynamics of each agent
- closely related to, but different from, the projective tensor product norm
  R. A. Ryan. *Introduction to Tensor Products of Banach Spaces*. Springer, 2002. ISBN 9781852334376

$(\Pi_n \otimes I_k)x$ measures **dissimilarity** of the states x_i:

$$x = \mathbb{1}_n \otimes x^* \quad \Longrightarrow \quad (\Pi_n \otimes I_k)x = (\Pi_n \otimes I_k)(\mathbb{1}_n \otimes x^*) = \Pi_n \mathbb{1}_n \otimes x^* = \mathbb{0}_{(n-1)\times k}$$

Example: Diffusively-coupled oscillators

$$\dot{x}_i = f(t, x_i) - \sum_{j=1}^{n} a_{ij}(x_i - x_j), \quad i \in \{1, \dots, n\}$$

G is an undirected weighted graph with Laplacian L

Suppose there exist $p \in [1, \infty]$, $Q \in \mathbb{R}^{k \times k}$

$$\mathrm{osL}_{p,Q}(f) < \lambda_2(L)$$

then semi-contraction wrt $\|\cdot\|_{(2,p),(\Pi_n \otimes Q)}$ and rate $c = \lambda_2(L) - \mathrm{osL}_{p,Q}(f)$,

1. for every trajectory $x(t)$,

$$\|x(t) - \mathbb{1}_n \otimes x_{\mathrm{ave}}(t)\|_{(2,p),(\Pi_n \otimes Q)} \; \leq \; e^{-ct}\|x(0) - \mathbb{1}_n \otimes x_{\mathrm{ave}}(0)\|_{(2,p),(\Pi_n \otimes Q)}$$

2. synchronization: $\lim_{t \to \infty} x(t) = \mathbb{1}_n \otimes x_{\mathrm{ave}}(t)$ where $x_{\mathrm{ave}}(t) = \frac{1}{n}\sum_{i=1}^{n} x_i(t)$

Example: Diffusively-coupled oscillators

$$\mathrm{osL}_{p,Q}(f) \leq \lambda_2(L)$$

- trade off between **internal dynamics** and **coupling strength**
- f time-invariant: every trajectory converges to the unique equilibrium point.
- f periodic: every trajectory converges to the unique periodic orbit.
- for any Lipschitz unstable dynamics f,
  there exists sufficiently strong coupling $\lambda_2(L)$ s.t. the network synchronizes.

Summary

- main equivalence theorem
  notion of one-sided Lipschitz constant and weak pairing
- characterization of contraction wrt non-Eulidean ℓ_1, ℓ_∞ norms
- robustness and iss properties
- network contraction theory
  - small-gain theorems
  - weak contraction
  - semi-contraction

Future work

1. forthcoming: fixed point algorithms and theorems
2. maybe: Halanay inequalities
3. maybe: optimal control problems for osL control systems

Incremental ISS for strongly contracting delay ODEs

$$\dot{x}(t) = f\big(x(t), x(t-s), u(t)\big), 0 \leq s \leq S, \qquad \|\cdot\|_{\mathcal{X}}, \|\cdot\|_{\mathcal{U}} \tag{13}$$

assume there exist positive constants c, $\ell_{\mathcal{U}}$, $\ell_{\mathcal{X}}$ such that, for all variables,

$$\textbf{osL } x: \quad [\![f(x,d,u) - f(y,d,u), x - y]\!]_{\mathcal{X}} \leq -c\|x - y\|_{\mathcal{X}}^2 \tag{14}$$

$$\textbf{Lip } x(t-s): \quad \|f(x,x_1,u) - f(x,x_2,u)\|_{\mathcal{X}} \leq \ell_{\mathcal{X}}\|x_1 - x_2\|_{\mathcal{X}} \tag{15}$$

$$\textbf{Lip } u: \quad \|f(x,d,u) - f(x,d,v)\|_{\mathcal{X}} \leq \ell_{\mathcal{U}}\|u - v\|_{\mathcal{U}} \tag{16}$$

By the curve norm derivative formula, subadditivity, and Cauchy-Schwarz inequality,

$$\begin{aligned}
\|x(t)-y(t)\|_{\mathcal{X}} D^+\|x(t) - y(t)\|_{\mathcal{X}} &= [\![f(x(t), x(t-s), u_x(t)) - f(y(t), y(t-s), u_y(t)), x(t) - y(t)]\!]_{\mathcal{X}} \\
&\leq [\![f(x(t), x(t-s), u_x(t)) - f(y(t), x(t-s), u_x(t)), x(t) - y(t)]\!]_{\mathcal{X}} \\
&\quad + [\![f(y(t), x(t-s), u_x(t)) - f(y(t), y(t-s), u_x(t)), x(t) - y(t)]\!]_{\mathcal{X}} \\
&\quad + [\![f(y(t), y(t-s), u_x(t)) - f(y(t), y(t-s), u_y(t)), x(t) - y(t)]\!]_{\mathcal{X}} \\
&\leq -c\|x(t) - y(t)\|_{\mathcal{X}}^2 + \ell_{\mathcal{X}}\|x(t-s) - y(t-s)\|_{\mathcal{U}}\|x(t) - y(t)\|_{\mathcal{X}}, \\
&\quad + \ell_{\mathcal{U}}\|u_x(t) - u_y(t)\|_{\mathcal{U}}\|x(t) - y(t)\|_{\mathcal{X}}.
\end{aligned}$$

Thus, with $m(t) = \|x(t) - y(t)\|_{\mathcal{X}}$, delay differential inequality:

$$D^+ m(t) \leq -cm(t) + \ell_{\mathcal{X}} \sup_{0 \leq s \leq S} m(t-s) + \ell_{\mathcal{U}}\|u_x(t) - u_y(t)\|_{\mathcal{U}}, \tag{17}$$

Halanay inequality is applicable. If $c > \ell_{\mathcal{X}}$, then

$$m(t) \leq m_0 \mathrm{e}^{-\rho(t-t_0)} + \ell_{\mathcal{U}} \int_{t_0}^{t} \mathrm{e}^{-\rho(t-\tau)}\|u_x(\tau) - u_y(\tau)\|_{\mathcal{U}} d\tau, \tag{18}$$

where $\rho > 0$ is the unique positive root of $\rho = c - \ell_{\mathcal{X}}\mathrm{e}^{\rho S}$ and $m_0 = \sup_{0 \leq s \leq S} m(t_0 - s)$.

Networks of contracting systems with time delays

Interconnected subsystems $i \in \{1, \dots, n\}$

$$\dot{x}_i = f_i(x_i, x_{-i}, x_{-i}(t-s), u_i), \qquad 0 \le s \le S, \qquad \|\cdot\|_i, \|\cdot\|_{i,\mathcal{U}} \qquad (19)$$

Assume there exist positive constants st

osL x_i : $\quad [\![f_i(x_i, \dots) - f_i(y_i, \dots), x_i - y_i]\!]_i \le -c_i \|x_i - y_i\|_i^2$

Lip x_{-i} : $\quad \|f_i(\dots, x_{-i}, \dots) - f_i(\dots, y_{-i}, \dots)\|_i \le \sum_{j=1, j\neq i}^{n} \gamma_{ij} \|x_j - y_j\|_j$

Lip x_{-1}^{-s} : $\quad \|f_i(\dots, x_{-i}^{-s}, \dots) - f_i(\dots, y_{-i}^{-s}, \dots)\|_i \le \sum_{j=1, j\neq i}^{n} \widehat{\gamma}_{ij} \|x_j^{-s} - y_j^{-s}\|_j$

Lip u_i : $\quad \|f_i(\dots, u_i) - f_i(\dots, v_i)\|_i \le \ell_{i,\mathcal{U}} \|u_i - v_i\|_{i,\mathcal{U}}$

With $m_i(t) = \|x_i(t) - y_i(t)\|_i$, delay differential inequality:

$$D^+ m(t) \le -C m(t) + \Gamma m(t) + \widehat{\Gamma} \sup_{0 \le s \le S} m(t-s) + \ell_{\mathcal{U}} \|u_x(t) - u_y(t)\|_{\mathcal{U}}$$

and, if the Metzler matrix $-C + \Gamma + \widehat{\Gamma}$ is Hurwitz, then (19) is incremental ISS

F. Mazenc, M. Malisoff, and M. Krstic. Vector extensions of Halanay's inequality. *IEEE Transactions on Automatic Control*, 2021. . to appear

Background on one-sided Lipschitz continuity

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