# Geometry, Analysis and Computation for Network Systems

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**Saber Jafarpour**
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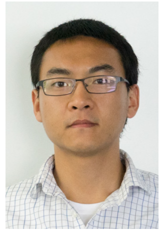

Xiaoming Duan
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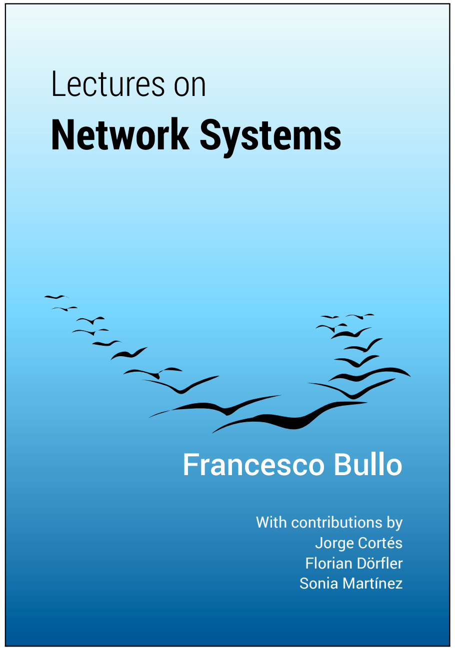
Kevin D. Smith
UCSB

NSF AFOSR ARO ONR DOE



## Lectures on Network Systems




**Lectures on Network Systems**, Francesco Bullo, Createspace, 1 edition, 2018, ISBN 978-1-986425-64-3

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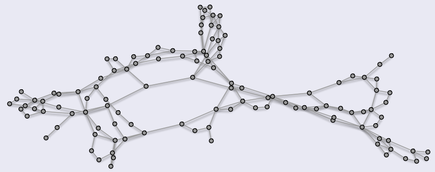
## Outline

1. **Linear Network Systems and Metzler Matrices**
   X. Duan, S. Jafarpour, and F. Bullo. Graph-theoretic small gain theorems for Metzler matrices and monotone systems.
   *IEEE Transactions on Automatic Control*, June 2019.
   Submitted.
   URL: https://arxiv.org/pdf/1905.05868.pdf
2. An emerging theory for Nonlinear Network Systems
3. Kuramoto Synchronization (existence and lack of uniqueness)

# Linear network systems

$$\dot{x}(t) = Ax(t)$$

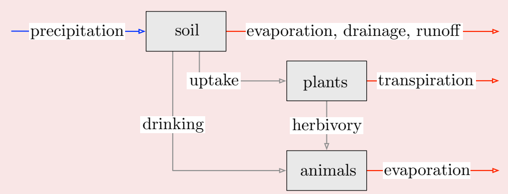


**network structure** $\iff$ **function = asymptotic behavior**

| Model | Dynamics | Asy Behavior | Graph property |
|---|---|---|---|
| averaging flow (Abelson '64) | $\dot{x} = -Lx$ Laplacian matrix | consensus | $\exists$ globally reach node |
| network flow (Noy Meir '73) | $\dot{x} = -L^\top x$ transpose Laplacian | stationary distribution | $\exists$ globally reach node |
| network flow with decay (outflows) | $\dot{x} = Cx$ $C = -L^\top - \mathrm{diag}(d)$ compartmental matrix | stability | outflow-connected |
| network flow with decay/growth | $\dot{x} = Mx$ $M = -L^\top + \mathrm{diag}(g-d)$ Metzler matrix | stability | **unknown** |



# Network flow systems




Water flow model for a desert ecosystem (Noy-Meir '73)

$$\dot{q}_i = \sum_j (F_{j\to i} - F_{i\to j}) - F_{i\to 0} + u_i$$

$$F_{i\to j} = f_{ij} q_i, \quad F = [f_{ij}]$$

$$\dot{q} = \underbrace{\left(F^T - \mathrm{diag}(F\mathbb{1}_n + f_0)\right)}_{=:C} q + u$$

$C$ **compartmental matrix**:
quasi-positive (off-diag $\geq 0$) and non-positive column sums ($f_0 \geq 0$)
analysis tools: *PF for quasi-positive, inverse positivity, algebraic graph*

system (= each condensed sink) is outflow-connected $\iff$ $C$ is Hurwitz

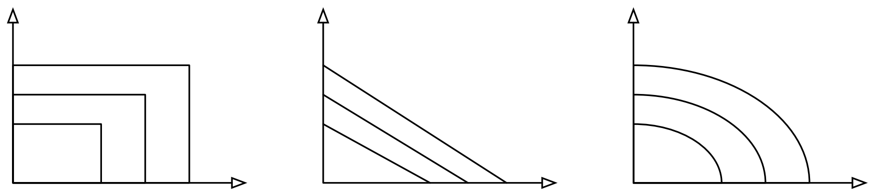
$\lim_{t\to\infty} q(t) = -C^{-1}u \geq 0$
$(-C^{-1}u)_i > 0 \iff$ $i$th compartment is inflow-connected



# Stability of network flow systems

A Metzler $M$ is Hurwitz iff any following equivalent condition hold:

1. there exists $\xi \in \mathbb{R}^n$ such that $\xi > \mathbb{0}_n$ and $M\xi < \mathbb{0}_n$;
2. there exists $\eta \in \mathbb{R}^n$ such that $\eta > \mathbb{0}_n$ and $\eta^\top M < \mathbb{0}_n^\top$; or
3. there exists a diagonal matrix $P \succ 0$ such that $M^\top P + PM \prec 0$.



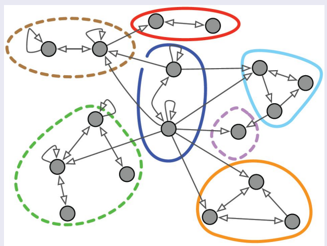
(a) $\max_{i\in\{1,\dots,n\}} x_i/\xi_i$ (b) $\eta^\top x$ (c) $x^\top Px$

**Goal: graph-theoretic conditions for stability**



# Reducible and acyclic graphs

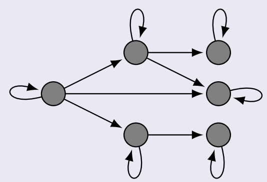
## Reducible graphs



$M \in \mathbb{R}^{n\times n}$ is Hurwitz
$\Updownarrow$
Strongly connected components are Hurwitz

Implication: large-scale system may be decomposed into smaller systems

## Directed acyclic graphs

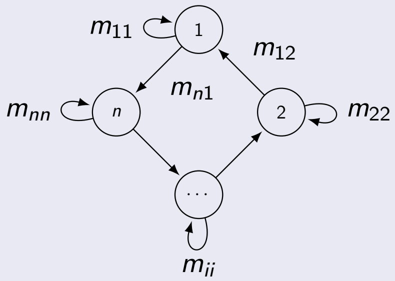


$M \in \mathbb{R}^{n\times n}$ is Hurwitz
$\Updownarrow$
diagonal entries are negative

Implication: study cycles!

# Basic ideas: a simple cycle

$$M = \begin{bmatrix} m_{11} & m_{12} & 0 & \cdots & 0 \\ 0 & m_{22} & m_{23} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & m_{n-1,n-1} & m_{n-1,n} \\ m_{n1} & 0 & \cdots & 0 & m_{nn} \end{bmatrix}$$




$$M \text{ Hurwitz} \iff \left(\frac{m_{12}}{-m_{11}}\right)\left(\frac{m_{23}}{-m_{22}}\right)\cdots\left(\frac{m_{n1}}{-m_{nn}}\right) < 1$$

where

- $\frac{m_{ij}}{-m_{ii}}$ represents a "gain" for subsystem $i$ with respect to $j$
- test: composition of "gains" along the cycle is less than 1



# Basic ideas: Small-gain network stability

**Cyclic Small-Gain Theorem**

a network of systems with input is ISS if

$$\text{cycle gain } < 1$$

about each simple cycle,
for appropriate interconnection gains


1. V. Lakshmikantham, V. M. Matrosov, and S. Sivasundaram. *Vector Lyapunov Functions and Stability Analysis of Nonlinear Systems.* Kluwer Academic Publishers, 1991
2. S. N. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. Small gain theorems for large scale systems and construction of ISS Lyapunov functions. *SIAM Journal on Control and Optimization*, 48(6):4089–4118, 2010. doi:10.1137/090746483
3. T. Liu, D. J. Hill, and Z.-P. Jiang. Lyapunov formulation of ISS cyclic-small-gain in continuous-time dynamical networks. Automatica, 47(9):2088–2093, 2011




# Summary of results

Thm 1: Input-to-state interconnection gains for Metzler systems

Thm 2: Max-interconnection gains and graph-theoretic conditions

Thm 3: Sum-interconnection gains and graph-theoretic conditions


X. Duan, S. Jafarpour, and F. Bullo. Graph-theoretic small gain theorems for Metzler matrices and monotone systems.
*IEEE Transactions on Automatic Control*, June 2019.
Submitted.
URL: https://arxiv.org/pdf/1905.05868.pdf




# Possible notions of ISS gains

An interconnected nonlinear system with subsystem dynamics

$$\dot{x}_i = f_i(x_i, x_{\mathcal{N}_i}, u_i), \qquad \forall i \in \{1, \dots, n\}.$$

system has sum-interconnection gains $\{\gamma_{ij}\}$ if

$$|x_i(t)| \;\leq\; \beta_i(|x_i(0)|, t) \;+\; \sum_{j \in \mathcal{N}_i} \gamma_{ij}(\|x_j\|_{[0,t]}) \;+\; \gamma_i(\|u_i\|_\infty).$$

where $\beta_i \in \mathcal{KL}$, $\gamma_{ij} \in \mathcal{K}$, and $\gamma_i \in \mathcal{K}$.

system has max-interconnection gains $\{\psi_{ij}\}$ if

$$|x_i(t)| \;\leq\; \max_{j \in \mathcal{N}_i} \{\beta'_i(|x_i(0)|, t),\; \psi_{ij}(\|x_j\|_{[0,t]}),\; \psi_i(\|u_i\|_\infty)\}.$$

where $\beta_i \in \mathcal{KL}$, $\psi_{ij} \in \mathcal{K}$, and $\psi_i \in \mathcal{K}$.

# Thm 1: ISS gains for Metzler systems

**Thm 1: ISS gains for Metzler systems**

For Metzler system $\dot{x} = Mx + u$, $M$ with negative diagonals,

1. sum-interconnection gains $\{\gamma_{ij}\}$ satisfy

$$\frac{m_{ij}}{-m_{ii}} \leq \gamma_{ij}, \quad \forall i \in \{1, \dots, n\}, j \in \mathcal{N}_i$$

2. max-interconnection gains $\{\psi_{ij}\}$ satisfy

$$\sum_{j \in \mathcal{N}_i} \left(\frac{m_{ij}}{-m_{ii}}\right) \psi_{ij}^{-1} < 1, \quad \forall i \in \{1, \dots, n\}$$

For $c = (i_1, i_2, \dots, i_k, i_1)$ be a simple cycle

1. the sum-cycle gain of $c$ is $\gamma_c = (\gamma_{i_2 i_1})(\gamma_{i_3 i_2}) \dots (\gamma_{i_1 i_k})$
2. a max-cycle gain of $c$ is $\psi_c = (\psi_{i_2 i_1})(\psi_{i_3 i_2}) \dots (\psi_{i_1 i_k})$



# Thm 2: Max-cycle gains and graph conditions

**Thm 2: Conditions based on max-cycle gains**

Given an irreducible Metzler matrix $M \in \mathbb{R}^{n \times n}$ with negative diagonal elements and the set of simple cycles $\Phi$, the followings are equivalent:

1. $M$ is Hurwitz;
2. for every $i \in V$ and $j \in \mathcal{N}_i$, there exists $\psi_{ij} > 0$ such that

$$\sum_{j \in \mathcal{N}_i} \left(\frac{m_{ij}}{-m_{ii}}\right) \psi_{ij}^{-1} < 1, \qquad \forall i \in \{1, \dots, n\},$$
$$\psi_c < 1, \qquad \forall c \in \Phi.$$

- "cycle gain $< 1$ about each simple cycle" is now IFF
- convex problem



# Thm 3: Sum-cycle gains and graph conditions

**Thm 3: Conditions based on sum-cycle gains**

Given an irreducible Metzler matrix $M \in \mathbb{R}^{n \times n}$ with negative diagonal elements, the followings are equivalent:

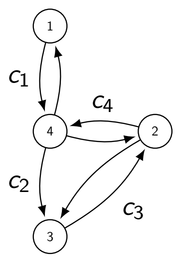
1. $M$ is Hurwitz;
2. for each $i$, let $\Phi_i$ be simple cycles over $\{1, \dots, i\}$ (or renumbered)

$$\sum_{c_1 \in \Phi_i} \gamma_{c_1} - \sum_{\substack{\{c_1, c_2\} \subset \Phi_i \\ c_1 \cap c_2 = \emptyset}} \gamma_{c_1} \gamma_{c_2} + \dots + \sum_{\substack{\{c_1, \dots, c_{r_i}\} \subset \Phi_i \\ c_i \cap c_j = \emptyset}} (-1)^{r_i - 1} \gamma_{c_1} \dots \gamma_{c_{r_i}} < 1$$

- condition 2 $\iff$ certain sums of products of gains $< 1$
- computation of sum-cycle gains and "sums of products" is straightforward (not iterative)



# Thm 3: Example




$$V_1 = \{1\} \implies \emptyset$$
$$V_2 = \{1, 4\} \implies \{\gamma_{c_1} < 1\}$$
$$V_3 = \{1, 4, 2\} \implies \{\gamma_{c_1} + \gamma_{c_4} < 1\}$$
$$V_4 = \{1, 4, 2, 3\} \implies \{\gamma_{c_1} + \gamma_{c_4} < 1, \; \gamma_{c_1} + \gamma_{c_2} + \gamma_{c_3} + \gamma_{c_4} - \gamma_{c_1}\gamma_{c_3} < 1\}$$

Hence, stability certificate

$$\gamma_{c_1} + \gamma_{c_4} < 1$$
$$\gamma_{c_1} + \gamma_{c_2} + \gamma_{c_3} + \gamma_{c_4} - \gamma_{c_1}\gamma_{c_3} < 1$$

# Outline

1. Linear Network Systems and Metzler Matrices

2. An emerging theory for Nonlinear Network Systems
   F. Bullo. *Lectures on Network Systems*.
   Kindle Direct Publishing, 1.3 edition, July 2019.
   With contributions by J. Cortés, F. Dörfler, and S. Martínez.
   URL: http://motion.me.ucsb.edu/book-lns

3. Kuramoto Synchronization (existence and lack of uniqueness)



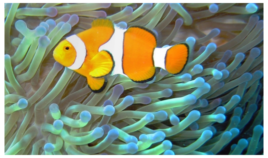
# Nonlinear network systems

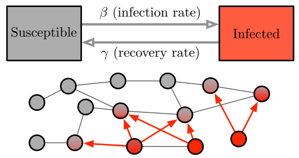
**Rich variety of emerging behaviors**

1. equilibria / limit cycles / extinction in populations dynamics
2. epidemic outbreaks in spreading processes
3. synchrony and multi-stability in coupled oscillators

**Rich variety of analysis tools**

1. nonlinear stability theory
2. passivity, small gain theorems, and dissipativity
3. contractivity and monotonicity

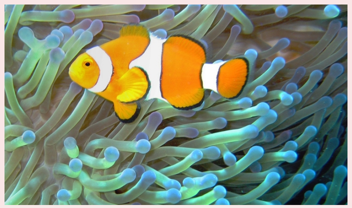







# Example: Population systems in ecology

(Vito Volterra, Universita' di Torino, 1860-1940)



Mutualism clownfish / anemones (Takeuchi et al '78)

Lotka-Volterra: $x_i$ = quantity/density

$$\frac{\dot{x}_i}{x_i} = b_i + \sum_j a_{ij} x_j$$

$$\dot{x} = \mathrm{diag}(x)(Ax + b)$$

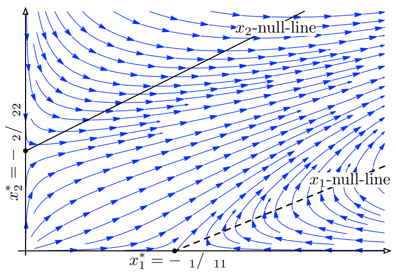
**interaction matrix** $A$:

$(+,+)$ mutualism, $(+,-)$ predation, $(-,-)$ competition
rich behavior: persistence, extinction, equilibria, periodic orbits, ...

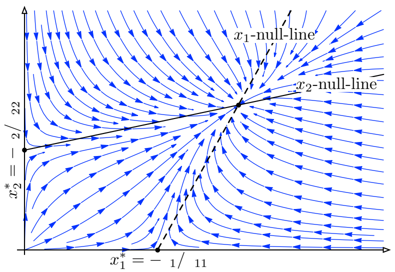
1. **mutualism:** $a_{ij} \geq 0$
2. either unbounded evolution or
   exists unique steady state $-A^{-1}b > 0$
   $\lim_{t\to\infty} x(t) = -A^{-1}b$ from all $x(0) > 0$



# Dichotomy in mutualistic Lotka-Volterra system




Case I: $a_{12} > 0$, $a_{21} > 0$, $a_{12}a_{21} > a_{11}a_{22}$. There exists no positive equilibrium point. All trajectories starting in $\mathbb{R}^2_{>0}$ diverge.




Case II: $a_{12} > 0$, $a_{21} > 0$, $a_{12}a_{21} < a_{11}a_{22}$. There exists a unique positive equilibrium point. All trajectories starting in $\mathbb{R}^2_{>0}$ converge to the equilibrium point.

## Research questions in Nonlinear Network Systems

1. what are key example systems?
2. what is a useful underlying structure?
3. what is a practical, simple, rich technical approach?
4. how do we treat dichotomy and richer behaviors?
5. how do we automatically generate Lyapunov functions?



## Example systems

**Kuramoto oscillators ('75)**

$$\dot\theta_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

Metzler Jac: phase cohesive region

Ex: active power flow, motion patterns

**Yorke network propagation ('76)**

$$\dot x = \beta\big(I_n - \operatorname{diag}(x)\big)Ax - \gamma x$$

Metzler Jac and positive

Ex: network SIR, patchy SIS

**Lotka-Volterra population ('20)**

$$\dot x = \operatorname{diag}(x)\big(Ax + r\big)$$

Metzler Jac: mutualistic interactions

Ex: biochemical networks, repressilator with 2 genes

**Daganzo cell transmission ('94)**

$$\dot\rho_e = f_e^{\text{in}}(\rho) - f_e^{\text{out}}(\rho)$$

Metzler Jac: free flow region

Ex: monotone distributed routing (Como, Savla, et al), Maeda '78, Sandberg '78

**Matrosov interconnection of ISS systems ('71)**

$$\dot x_i = f_i(x_1, \dots, x_n, u_i) \quad \Longrightarrow \quad \dot v \leq -A(v) + \Gamma(v) + G(w)$$

Metzler Jac and positive



## A review of Contraction Theory

given norm, the **matrix measure** of $A$ is

$$\mu(A) := \lim_{h \to 0^+} \frac{\|I_n + hA\| - 1}{h}$$

assume: vector field $f$ is **infinitesimally contracting** over $C$, that is,

$$\mu\big(Df(x)\big) \leq c < 0, \quad \text{for all } x \in C$$

assume: set $C$ is ***f*-invariant**, closed and convex

**Desirable consequences**

1. flow of $f$ is a contraction, i.e., distance between solutions exponentially decreases with rate $c$
2. there exists an equilibrium $x^*$, unique, globally exponentially stable with global Lyapunov functions

$$x \mapsto \|x - x^*\|^2 \quad \text{and} \quad x \mapsto \|f(x)\|^2$$



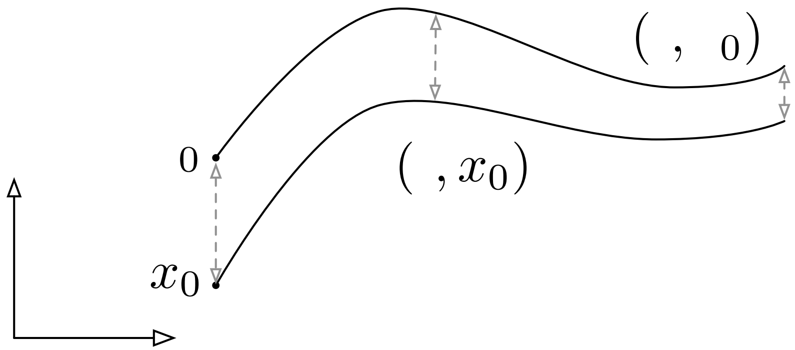



Figure: Any two trajectories of an infinitesimally contracting system converge.

## Common matrix measures

| Vector norm | Matrix measure |
|---|---|
| $\|x\|_1 = \sum_{i=1}^n \|x_i\|$ | $\mu_1(A) = \max_{j\in\{1,\dots,n\}} \left(a_{jj} + \sum_{i=1,i\neq j}^n \|a_{ij}\|\right)$ $=$ max column "absolute sum" of $A$ |
| $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$ | $\mu_2(A) = \lambda_{\max}\left(\frac{A+A^\top}{2}\right)$ |
| $\|x\|_\infty = \max_{i\in\{1,\dots,n\}} \|x_i\|$ | $\mu_\infty(A) = \max_{i\in\{1,\dots,n\}} \left(a_{ii} + \sum_{j=1,j\neq i}^n \|a_{ij}\|\right)$ $=$ max row "absolute sum" of $A$ |

Simplifications for a Metzler matrix $M$

$$\mu_1(M) = \max_{j\in\{1,\dots,n\}} \sum_{i=1}^n m_{ij} = \max(M^\top \mathbb{1}_n) = \text{max column sum of } M$$

$$\mu_\infty(M) = \max_{i\in\{1,\dots,n\}} \sum_{j=1}^n m_{ij} = \max(M\mathbb{1}_n) = \text{max row sum of } M$$



## The Euclidean case: works by Krasovskiĭ & Vidyasagar

**Vidyasagar '78: Lyapunov functions and matrix measures**

Given $P \succ 0$ and $c \in \mathbb{R}$,

$$\mu_{2,P}(A) < c \qquad \iff \qquad A^\top P + PA \prec 2cP$$

1. $A$ Hurwitz $\iff$ $A$ has negative weighted 2-norm (w.r.t. some $P$)
2. $\inf_{P\succ 0} \mu_{2,P}(A) =$ spectral abscissa of $A$

**Krasovskiĭ '60: method to design Lyapunov function**

$f$ is weighted 2-norm contracting if $\exists P \succ 0$ and $c < 0$

$$P\,Df(x) + Df(x)^\top P \preceq 2cP, \qquad \text{for all } x \in \mathbb{R}^n$$

Constant Lyapunov weight $P$ at each $x$ implies desirable consequences



## The non-Euclidean case for Metzler Jacobians

**Coogan '16: matrix measures of a Metzler matrix $M$**

Given vectors $\eta, \xi > \mathbb{0}_m$ and $c \in \mathbb{R}$,

$$\mu_{1,\mathrm{diag}(\eta)}(M) < c \iff \eta^\top M < c\eta^\top, \text{ and}$$

$$\mu_{\infty,\mathrm{diag}(\xi)^{-1}}(M) < c \iff M\xi < c\xi,$$

1. $M$ Hurwitz $\iff$ $M$ has negative weighted 1- or $\infty$-measure
2. $\inf_{\eta>\mathbb{0}_m} \mu_{1,\mathrm{diag}(\eta)}(M) = \inf_{\xi\mathbb{0}_m} \mu_{\infty,\mathrm{diag}(\xi)^{-1}}(M) =$ spectral abscissa of $M$

**Sum-separable and max-separable Lyapunov functions**

$f$ with Metzler Jac is weighted 1-norm contracting if $\exists \eta > \mathbb{0}_n$ and $c < 0$

$$\eta^\top Df(x) \leq c\eta^\top, \qquad \text{for all } x \in \mathbb{R}^n$$

Constant column weights $\eta$ at each $x$ implies desirable consequences



## Krasovskiĭ Lyapunov functions for systems with Metzler Jacobians and constant weights

**Weighted diagonal 2-norm:**

$$\|x - x^*\|_P^2 = \sum_{i=1}^n p_i (x_i - x_i)^2 \quad \text{and} \quad \|f(x)\|_P^2 = \sum_{i=1}^n p_i f_i(x)^2$$

**Weighted 1-norm**

$$\|x - x^*\|_{1,\eta} = \sum_{i=1}^n \eta_i |x_i - x_i^*| \quad \text{and} \quad \|f(x)\|_{1,\eta} = \sum_{i=1}^n \eta_i |f_i(x)|$$

**Weighted $\infty$-norm**

$$\|x - x^*\|_{\infty,\xi^{-1}} = \max_{i\in\{1,\dots,n\}} \frac{|x_i - x_i^*|}{\xi_i} \quad \text{and} \quad \|f(x)\|_{\infty,\xi^{-1}} = \max_{i\in\{1,\dots,n\}} \frac{|f_i(x)|}{\xi_i}$$

**Recall: sublevel sets of Lyapunov functions are $f$-invariant**

## Example application to Lotka-Volterra

1. change of variable $y = \ln x$, so that $x \in \mathbb{R}^n_{>0}$ maps into $y \in \mathbb{R}^n$ and

$$\dot{y} = A \exp(y) + r := f_{\mathrm{LVe}}(y)$$

2. pick $v > \mathbb{0}_n$ such that $v^\top A < \mathbb{0}_n$ and show

$$v^\top Df_{\mathrm{LVe}}(y) = v^\top A \operatorname{diag}(\exp(y)) < -cv^\top \operatorname{diag}(\exp(y)) \leq 0.$$

3. $f_{\mathrm{LVe}}$, and so $f_{\mathrm{LV}}$, has a unique globally exponentially stable equilibrium with sum-separable global Lyapunov functions

$$\|y - y^*\|_{1,\operatorname{diag}(v)} \quad \text{and} \quad \|f_{\mathrm{LVe}}(y)\|_{1,\operatorname{diag}(v)}$$

that is,

$$x \mapsto \sum_{i=1}^n v_i |\ln(x_i/x_i^*)|, \qquad x \mapsto \sum_{i=1}^n v_i |(Ax + r)_i|$$



## Weakly contracting systems

For a vector field $f$ a and norm

C1 there exists a convex and $f$-invariant set $C$,

C2 $f$ is infinitesimally weakly contractive on the set $C$

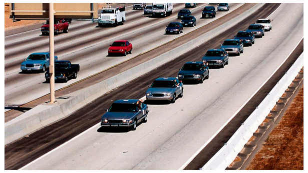
**Desirable consequences (under additional incremental assumptions)**

Then one of the following mutually-exclusive conditions hold: either

1. $f$ has no equilibrium in $C$ and every trajectory in $C$ is unbounded, or
2. $f$ has at least one equilibrium $x^* \in C$ and:
   1. every trajectory starting in $C$ is bounded and each equilibrium $x^{**}$ is stable with weak Lyapunov function $x \mapsto \|x - x^{**}\|$,
   2. if the norm $\|\cdot\|$ is a $(p, R)$-norm, $p \in \{1, \infty\}$ and $f$ is piecewise real analytic, then every trajectory converges to the set of equilibria,
   3. if $x^*$ is locally asy stable, then $x^*$ is globally asy stable in $C$,
   4. if $\mu(Df(x^*)) < 0$, then $x \mapsto \|x - x^*\|$ is a global Lyapunov function and $x \mapsto \|f(x)\|$ is a local Lyapunov function.



## Why is this relevant for infrastructure networks?




Consider a network flow system $\dot{x} = f(x)$ preserving a commodity

$$\begin{aligned} \text{constant} &= \mathbb{1}_n^\top x(t) \\ \implies 0 &= \mathbb{1}_n^\top \dot{x}(t) = \mathbb{1}_n^\top f(x(t)) \\ \implies \mathbb{0}_n &= \mathbb{1}_n^\top Df x(t) \end{aligned}$$

If additionally $f$ has Meztler Jacobian, then $f$ is automatically weakly contracting (non-expansive) with respect to the $\ell_1$ norm.



## Outline

1. Linear Network Systems and Metzler Matrices
2. An emerging theory for Nonlinear Network Systems
3. Kuramoto Synchronization (existence)
   S. Jafarpour and F. Bullo. Synchronization of Kuramoto oscillators via cutset projections. *IEEE Transactions on Automatic Control*, 64(7):2830–2844, 2019. doi:10.1109/TAC.2018.2876786
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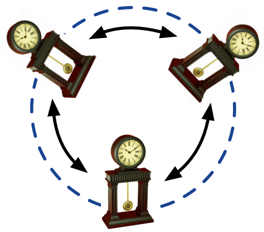
   URL: https://arxiv.org/pdf/1901.11189.pdf

# Today: Sync & Multi-Stability in Coupled Oscillators

**Kuramoto model**

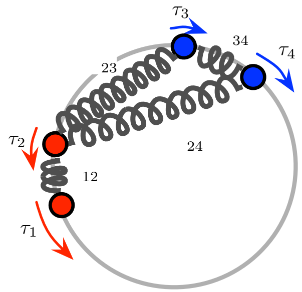
- $n$ **oscillators** with angle $\theta_i \in \mathbb{S}^1$
- **non-identical** natural frequencies $\omega_i \in \mathbb{R}^1$
- **coupling** with strength $a_{ij} = a_{ji}$

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$





# Model #1: Spring network analog and applications




**Coupled swing equations**

Euler-Lagrange eq for spring network on ring:

$$m_i \ddot{\theta}_i + d_i \dot{\theta}_i = \tau_i - \sum_j k_{ij} \sin(\theta_i - \theta_j)$$

**Kuramoto coupled oscillators**

$$\dot{\theta}_i = \omega_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$$

**Kuramoto equilibrium equation**

$$0 = \omega_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$$

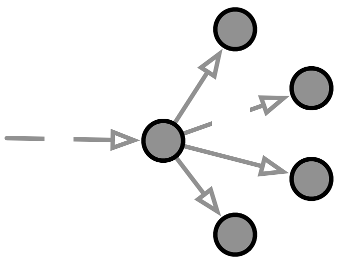


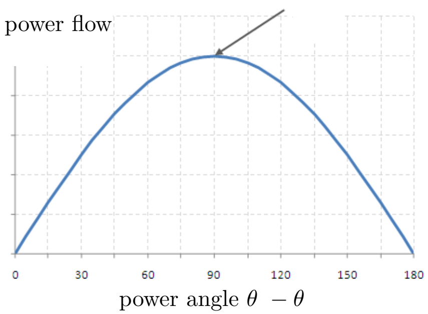
# Model #2: Active Power Flow Problem

AC, Kirckhoff and Ohm, quasi-sync, lossless lines, constant voltages.

supply/demand $p_i$, max power coeff $a_{ij}$, voltage phase $\theta_i$

$$p_i = \sum_{j=1}^{n} f_{ij}, \qquad f_{ij} = a_{ij} \sin(\theta_i - \theta_j)$$





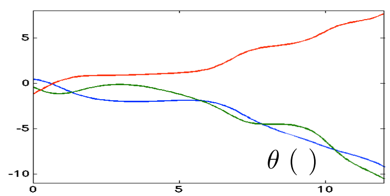

**Given:** network parameters & topology, load & generation profile,



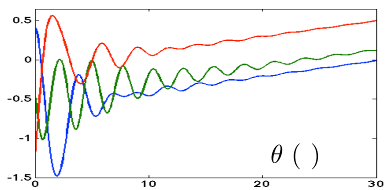
# Phenomenon #1: Transition from incoherence to sync

Function = synchronization

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$




large $|\omega_i - \omega_j|$ & small coupling
$\Rightarrow$ incoherence = no sync




small $|\omega_i - \omega_j|$ & large coupling
$\Rightarrow$ coherence = frequency sync

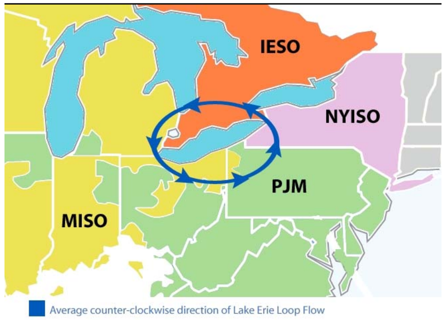
# Phenomenon #2: Multiple power flows

**Theoretical observation: multiple solutions exist**

Practical observations:

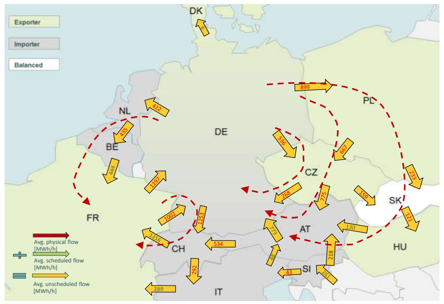
sometimes undesirable power flows around loops

sometimes sizable difference between predicted and actual power flows




New York Independent System Operator, **Lake Erie Loop Flow Mitigation**, Technical Report, 2008

Figure 8: Average unscheduled flows for the years 2011 and 2012, MWh/h




Source: THEMA Consulting Group, based on data from 16 TSOs

THEMA Consulting Group, **Loop-flows - Final advice**, Technical Report prepared for the European Commission, 2013



# Outline

1. Linear Network Systems and Metzler Matrices
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# Primer on algebraic graph theory (slide 1/2)

Weighted undirected graph with $n$ nodes and $m$ edges:

**Incidence matrix**: $n \times m$ matrix $B$ s.t. $(B^\top p_{\text{actv}})_{(ij)} = p_i - p_j$

**Weight matrix**: $m \times m$ diagonal matrix $\mathcal{A}$

**Laplacian stiffness**: $L = B\mathcal{A}B^\top \geq 0$

**Linearization of Kuramoto equilibrium equation**:

$$p_{\text{actv}} = B\mathcal{A}\sin(B^\top\theta) \quad \Longrightarrow \quad p_{\text{actv}} \approx B\mathcal{A}(B^\top\theta) = L\theta$$

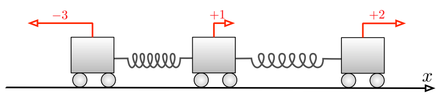
**Algebraic connectivity**:

$$\lambda_2(L) = \text{second smallest eig of } L$$
$$= \text{notion of connectivity and coupling}$$

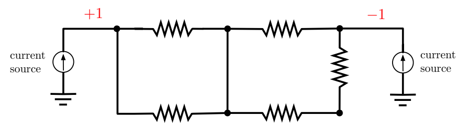


# Primer on algebraic graph theory (slide 2/2)

## Laplacian linear balance equation




(a) spring network




(b) resistive circuit

$$L_{\text{stiffness}}\, x = f_{\text{load}} \qquad \text{and} \qquad L_{\text{conductance}}\, v = c_{\text{injected}}$$

**Laplacian linear balance equation: $p_{\text{actv}} = L\theta$**

if $\sum_i p_i = 0$ in $p_{\text{actv}} = L\theta$, then equilibrium exists : $\theta = L^\dagger p_{\text{actv}}$

pairwise displacements : $B^\top\theta = B^\top L^\dagger p_{\text{actv}}$

## From Old to New Tests

Question: Given balanced $p_{\text{actv}}$, do angles exist satisfying

$$p_{\text{actv}} = B\mathcal{A}\sin(B^\top\theta)$$

Old Tests: Equilibrium angles (neighbors within $\pi/2$ arc) exist if

$$\|B^\top p_{\text{actv}}\|_2 < \lambda_2(L) \quad \text{for unweighted graphs} \quad \text{(Old 2-norm T)}$$
$$\|B^\top L^\dagger p_{\text{actv}}\|_\infty < 1 \quad \text{for trees, complete} \quad \text{(Old } \infty\text{-norm T)}$$

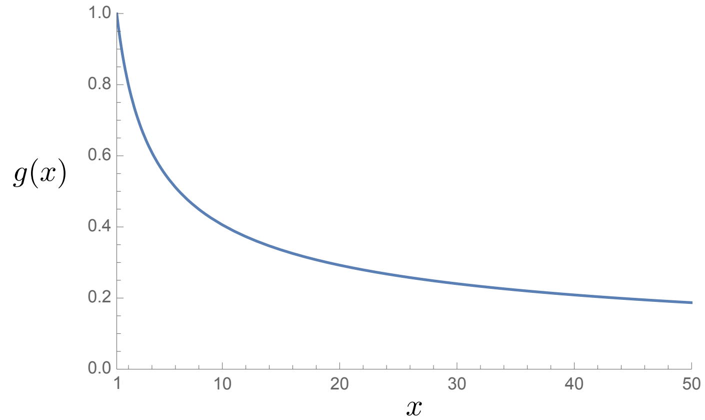
New Tests: Equilibrium angles (neighbors within $\pi/2$ arc) exist if

$$\|B^\top L^\dagger p_{\text{actv}}\|_2 < 1 \quad \text{for unweighted graphs} \quad \text{(New 2-norm T)}$$
$$\|B^\top L^\dagger p_{\text{actv}}\|_\infty < g(\|\mathcal{P}\|_\infty) \quad \text{for all graphs} \quad \text{(New } \infty\text{-norm T)}$$



## where $g$ is monotonically decreasing

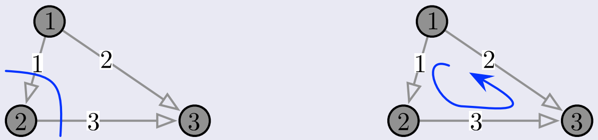
$$g : [1,\infty) \to [0,1]$$
$$x \mapsto \frac{y(x)+\sin(y(x))}{2} - x\,\frac{y(x)-\sin(y(x))}{2}\Big|_{y(x)=\arccos(\frac{x-1}{x+1})}$$






## and where $\mathcal{P}$ is a projection matrix

$$\mathcal{P} = B^\top L^\dagger B\mathcal{A} \quad = \text{oblique projection onto } \mathrm{Im}(B^\top) \text{ parallel to } \mathrm{Ker}(B\mathcal{A})$$




$$\underbrace{\mathbb{R}^m}_{\text{edge space}} = \underbrace{\mathrm{Im}(B^\top)}_{\substack{\text{cutset space}\\ \text{flow vectors}}} \oplus \underbrace{\mathrm{Ker}(B\mathcal{A})}_{\substack{\text{weighted cycle space}\\ \text{cycle vectors}}}$$

1. if $G$ unweighted, then $\mathcal{P}$ is orthogonal and $\|\mathcal{P}\|_2 = 1$
2. if $G$ acyclic, then $\mathcal{P} = I_m$ and $\|\mathcal{P}\|_p = 1$
3. if $G$ uniform complete or ring, then $\|\mathcal{P}\|_\infty = 2(n-1)/n \le 2$



New Tests: Equilibrium angles (neighbors within $\pi/2$ arc) exist if

$$\|B^\top L^\dagger p_{\text{actv}}\|_2 < 1 \quad \text{for unweighted graphs} \quad \text{(New 2-norm T)}$$
$$\|B^\top L^\dagger p_{\text{actv}}\|_\infty < g(\|\mathcal{P}\|_\infty) \quad \text{for all graphs} \quad \text{(New } \infty\text{-norm T)}$$

### Unifying theorem with a family of tests

Equilibrium angles (neighbors within $\gamma$ arc) exist if, in some $p$-norm,

$$\|B^\top L^\dagger p_{\text{actv}}\|_p \le \gamma\alpha_p(\gamma) \quad \text{for all graphs} \quad \text{(New } \alpha_p \text{ T)}$$

where nonconvex optimization problem:

$$\alpha_p(\gamma) := \text{ min amplification factor of } \quad \mathcal{P}\,\mathrm{diag}[\mathrm{sinc}(x)]$$

## Proof sketch 1/2: Rewriting the equilibrium equation

For what $B, \mathcal{A}, p_{\text{actv}}$ does there exist $\theta$ solution to:

$$p_{\text{actv}} = B\mathcal{A}\sin(B^\top\theta)$$

**STEP 1:** For what flow $z$ and projection $\mathcal{P}$ onto cutset/flow space, does there exist a flow $x$ that solves

$$\mathcal{P}\sin(x) = z$$

$$\iff \quad \mathcal{P}\,\text{diag}[\text{sinc}(x)]x = z$$

$$\iff \quad x = (\mathcal{P}\,\text{diag}[\text{sinc}(x)])^{-1}z =: h(x)$$



## Proof sketch 2/2: Amplification factor & Brouwer

**STEP 1:** look for $x$ solving

$$x = h(x) = (\mathcal{P}\,\text{diag}[\text{sinc}(x)])^{-1}z$$

**IDEA:** assume $\|x\|_p \leq \gamma$ and ensure $\|h(x)\|_p \leq \gamma$

**STEP 2:** if one defines min amplification factor

$$\alpha_p(\gamma) := \min_{\|x\|_p\leq\gamma} \min_{\|y\|_p=1} \|\mathcal{P}\,\text{diag}[\text{sinc}(x)]y\|_p$$

then $$\|h(x)\|_p \leq \max_x \max_y \|(\mathcal{P}\,\text{diag}[\text{sinc}(x)])^{-1}y\|_p \cdot \|z\|_p$$

$$= \big(\min_x \min_y \|\mathcal{P}\,\text{diag}[\text{sinc}(x)]y\|_p\big)^{-1}\|z\|_p \quad \leq \frac{\|z\|_p}{\alpha_p(\gamma)}$$

**STEP 3:** $\|z\|_p \leq \gamma\alpha_p(\gamma)$, then $\|h(x)\|_p \leq \gamma$ so that $h$ satisfies Brouwer



## Comparison of sufficient and approximate sync tests

Any test predicts max transmittable power (before bifurcation).
Compare with numerically computed.

| Test Case | ratio of test prediction to numerical computation | | | |
|---|---|---|---|---|
| | old 2-norm | new $\infty$-norm | $g(\|\mathcal{P}\|_\infty) \approx 1$ approximate | $\alpha_\infty$ test *fmincon* |
| IEEE 9 | 16.54 % | 73.74 % | 92.13 % | 85.06 %† |
| IEEE 14 | 8.33 % | 59.42 % | 83.09 % | 81.32 %† |
| IEEE RTS 24 | 3.86 % | 53.44 % | 89.48 % | 89.48 %† |
| IEEE 30 | 2.70 % | 55.70 % | 85.54 % | 85.54 %† |
| IEEE 118 | 0.29 % | 43.70 % | 85.95 % | —* |
| IEEE 300 | 0.20 % | 40.33 % | 99.80 % | —* |
| Polish 2383 | 0.11 % | 29.08 % | 82.85 % | —* |

† *fmincon* with 100 randomized initial conditions
* *fmincon* does not converge



## Summary: Kuramoto equilibrium and active power flow

Given topology (incidence $B$), admittances (Laplacian $L$), injections $p_{\text{actv}}$,

$$p_i = \sum_{j=1}^n a_{ij}\sin(\theta_i - \theta_j)$$

Equilibrium angles exist if, in some $p$-norm,

$$\|B^\top L^\dagger p_{\text{actv}}\|_p \leq \gamma\alpha_p(\gamma) \qquad \text{for all graphs} \qquad (\text{New } \alpha_p \text{ T})$$

For $p = \infty$, after bounding,

$$\|B^\top L^\dagger p_{\text{actv}}\|_\infty \leq g(\|\mathcal{P}\|_\infty) \qquad (\text{New } \infty\text{-norm T})$$

**Q1:** $\exists$ a **stable operating point** (with pairwise angles $\leq \gamma$)?
**Q2:** what is the **network capacity** to transmit active power?
**Q3:** how to quantify **robustness** as distance from loss of feasibility?

## Outline

1. Introduction to Network Systems
   F. Bullo. *Lectures on Network Systems*.
   Kindle Direct Publishing, 1.3 edition, July 2019.
   With contributions by J. Cortés, F. Dörfler, and S. Martínez.
   URL: http://motion.me.ucsb.edu/book-lns

2. Synchronization (existence)
   S. Jafarpour and F. Bullo. Synchronization of Kuramoto oscillators via cutset projections.
   *IEEE Transactions on Automatic Control*, 64(7):2830–2844, 2019.
   doi:10.1109/TAC.2018.2876786

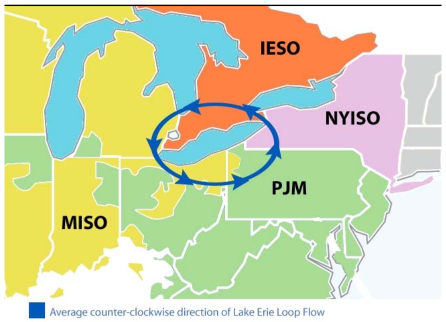
3. Multi-Stability (lack of uniqueness)
   S. Jafarpour, E. Y. Huang, K. D. Smith, and F. Bullo. Multistable synchronous power flows: From geometry to analysis and computation.
   *SIAM Review*, January 2019.
   Submitted.
   URL: https://arxiv.org/pdf/1901.11189.pdf



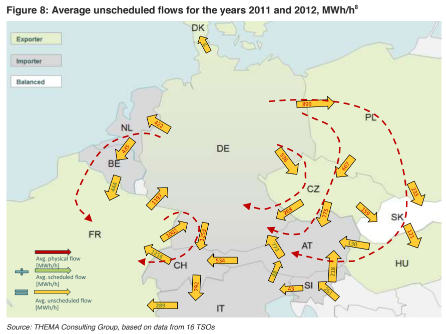
## Phenomenon #2: Multiple power flows

**Theoretical observation: multiple solutions exist**

Practical observations:
- sometimes undesirable power flows around loops
- sometimes sizable difference between predicted and actual power flows




New York Independent System Operator, Lake Erie Loop Flow Mitigation, Technical Report, 2008




THEMA Consulting Group, Loop-flows - Final advice, Technical Report prepared for the European Commission, 2013



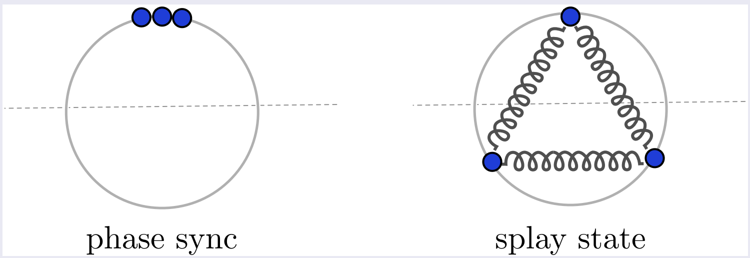
## Lack of uniqueness and winding solutions

Given topology (incidence $B$), admittances (Laplacian $L$), injections $p_{\text{actv}}$,

$$p_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

1. is solution unique?
2. how to localize/classify solutions?

triangle graph, homogeneous weights ($a_{ij} = 1$), $p_{\text{actv}} = 0$



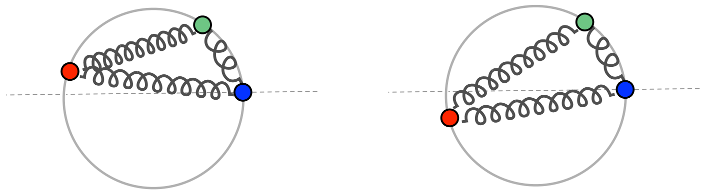



## Winding number of $n$ angles

Given undirected graph with a cycle $\sigma = (1, \dots, n_\sigma)$ and orientation

1. **winding number of $\theta \in \mathbb{T}^n$ along $\sigma$** is:

$$w_\sigma(\theta) = \frac{1}{2\pi} \sum_{i=1}^{n_\sigma} d_{cc}(\theta_i, \theta_{i+1})$$



$(\theta) = 0$ $\qquad$ $(\theta) = \pm 1$

2. given basis $\sigma_1, \dots, \sigma_r$ for cycles, **winding vector of $\theta$** is

$$w(\theta) = (w_{\sigma_1}(\theta), \dots, w_{\sigma_r}(\theta))$$

## "Kirckhoff Angle Law" and partition of the $n$-torus

**Theorem: Kirchhoff angle law on $\mathbb{T}^n$**

$$w_\sigma(\theta) = 0, \pm 1, \dots, \pm \lfloor n_\sigma/2 \rfloor$$
$$\implies w(\theta) \text{ is piecewise constant}$$
$$\implies w(\theta) \text{ takes value in a finite set}$$

**Theorem: Winding partition**

For each possible winding vector $u$, define

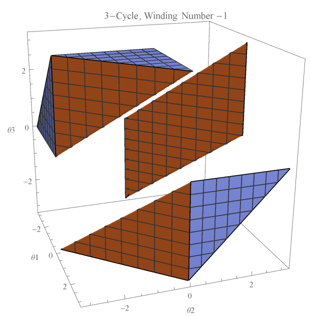
$$\mathrm{WindingCell}(u) := \{\theta \in \mathbb{T}^n \mid w(\theta) = u\}$$

Then

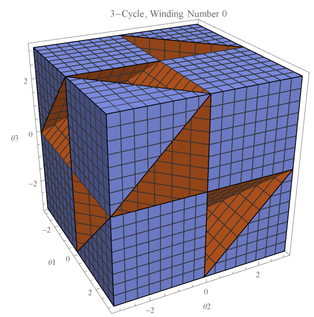
$$\mathbb{T}^n = \cup_u \mathrm{WindingCell}(u)$$



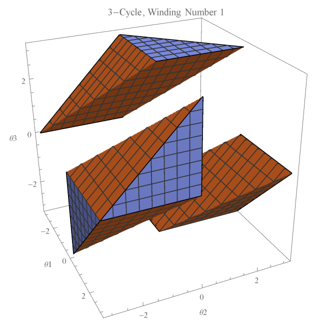
## Winding partition of triangle graph




$w = -1$

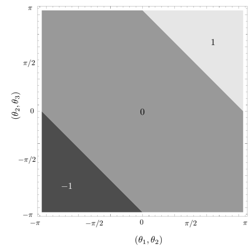



$w = 0$




$w = +1$

- each winding cell is connected
- each winding cell is invariant under rotation
- bijection:
  reduced winding cell $\longleftrightarrow$ open convex polytope

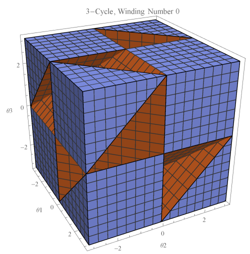





## The Kuramoto model and the winding partition

Given topology (incidence $B$), admittances (Laplacian $L$), injections $p_{\mathrm{actv}}$,

$$\dot{\theta}_i = p_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$$



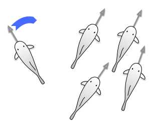

**Theorem: At-most-uniqueness and extensions**

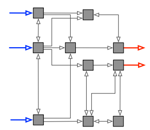
1. each WindingCell has at-most-unique equilibrium with $\Delta\theta < \pi/2$
2. equilibrium loop flow increases monotonically wrt winding number
3. existence + uniqueness in WindingCell($u$) with $\Delta\theta < \pi/2$ if

$$\|B^\top L^\dagger p_{\mathrm{actv}} + Cu\|_\infty \leq g(\|\mathcal{P}\|_\infty), \text{ or} \qquad \text{(Static T)}$$
$$\exists \text{ a trajectory inside WindingCell}(u) \text{ with } \Delta\theta < \pi/2 \qquad \text{(Dynamic T)}$$

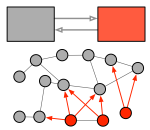


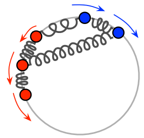
## Summary and Future Work

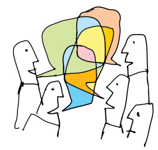

averaging


compartmental flows


mutualism


virus spread


coupled oscillators


social systems

### Review

1. a rather comprehensive theory of linear network systems
2. an emergent theory of nonlinear network systems based on contractivity and monotonicity
3. existence and multistability for Kuramoto

### Future research

1. a little bit more on Metzler matrices
2. much work on monotonicity and contractivity
3. applications to other dynamic flow networks
4. **outreach/collaboration opportunities for our community with**
   sociologists, biologists, economists, physicists ...