Network Systems
Theory and Applications to Synchronous Power Flows

Francesco Bullo

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2nd Colloquium Roberto Tempo on Automatica

CNR and Politecnico di Torino, Turin, Italy, Apr 12, 2019

In Roberto's honor

- Colloquia Roberto Tempo on Automatica, CNR, Turin, Italy, 2018
- PReGio Roberto Tempo Award, IEIIT-CNR, Italy, 2017
- IEEE CDC Roberto Tempo Best Paper Award, IEEE Control Systems Society, 2019

- Plenary session "A Tribute in Memory of Roberto Tempo", IFAC World Congress, Toulouse, France, Jul 2017 (Youtube link)
- "Scaling Heights: Our Times Shared with Roberto Tempo," Plenary special session and technical tutorial session, IEEE Conference in Decision and Control, Melbourne Australia, Dec 2017

- Book: "Uncertainty in Complex Networked Systems: In Honor of Roberto Tempo" editor T. Başar, Springer, 2018

- and many others

During his service to CSS

CSS ExCom trip, May 2011, Maynooth and Dublin, Ireland

Roberto's visits to UCSB

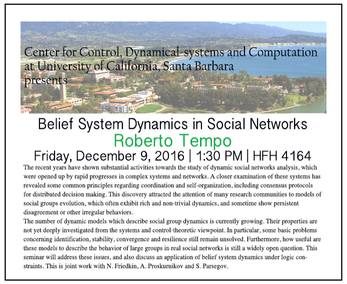
08nov10 "Design of Uncertain Complex Systems: A Randomization Viewpoint"

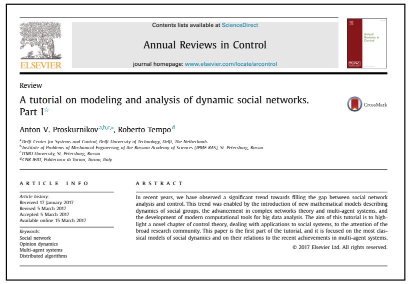
15nov11 "Information-based Complexity for Systems and Control: The Probabilistic Setting"

21oct14 "The PageRank Problem in Google: A Systems and Control Viewpoint"

22oct14 "Distributed Randomized Algorithms in Social and Sensor Networks"

09dec16 **"Belief System Dynamics in Social Networks"**



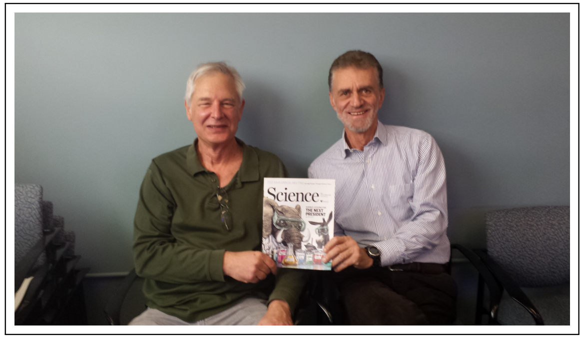


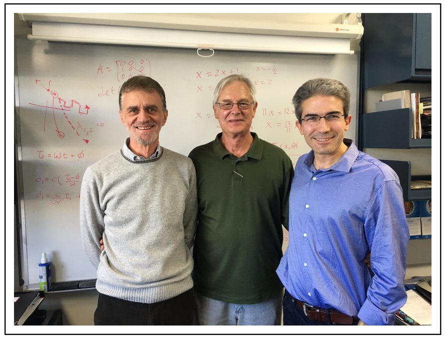

GROUP DYNAMICS

Network science on belief system dynamics under logic constraints

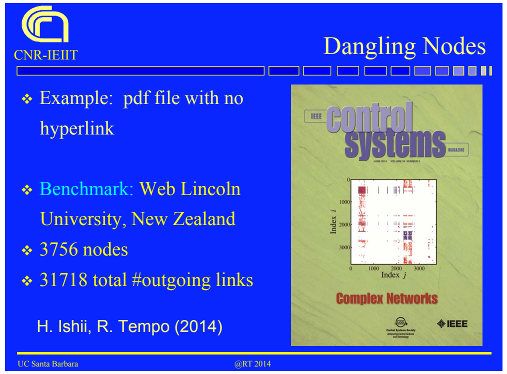
Noah E. Friedkin,[1*] Anton V. Proskurnikov,[2,3] Roberto Tempo,[4] Sergey E. Parsegov[5]

Breakthroughs have been made in algorithmic approaches to understanding how individuals in a group influence each other to reach a consensus. However, what happens to the group consensus if it depends on several statements, one of which is proven false? Here, we show how the existence of logical constraints on beliefs affect the collective convergence to a shared belief system and, in contrast, how an idiosyncratic set of arbitrarily linked beliefs held by a few may become held by many.





Roberto and network systems



Torino is now a worldwide leading center on network systems, with contributions by Anton Proskurnikov, Chiara Ravazzi, Fabio Fagnani, Fabrizio Dabbene, Francesca Ceragioli, Giacomo Como, Giuseppe Calafiore, Paolo Frasca, . . .

I imagine Roberto would be glad to hear us talk about these topics today

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IEEE CSS Italy — Control Systems Society Italy Chapter
IEEE PES — Power & Energy Society® Italy Chapter - PE31
Robotics & Automation Society - Italy Chapter
IEEE PoliTO Student Branch

Acknowledgments

Gregory Toussaint	Ketan Savla	Gábor Orosz*	Fabio Pasqualetti
Todd Cerven	Kurt Plarre*	Shaunak Bopardikar	A. Mirtabatabaei
Jorge Cortés*	Ruggero Carli*	Karl Obermeyer	Rush Patel
Sonia Martínez*	Nikolaj Nordkvist	Sandra Dandach	Pushkarini Agharkar
G. Notarstefano	Sara Susca	Joey Durham	Jeff Peters
Anurag Ganguli	Stephen Smith	Vaibhav Srivastava	Wenjun Mei

Saber Jafarpour
UCSB

Elizabeth Y. Huang
UCSB

Kevin D. Smith
UCSB

NSF AFOSR ARO ONR DOE

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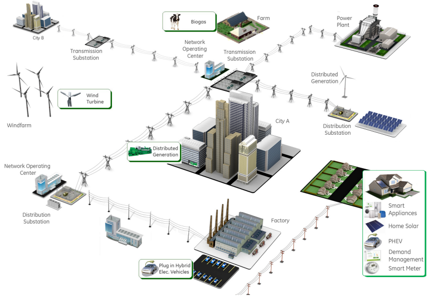
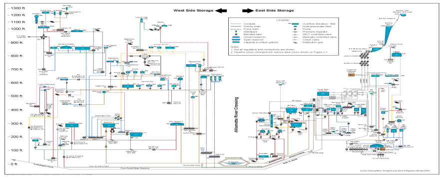
Network systems in technology



Smart grid

Amazon robotic warehouse



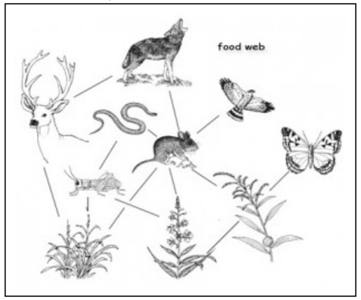
Portland water network

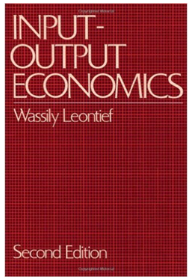
Industrial chemical plant

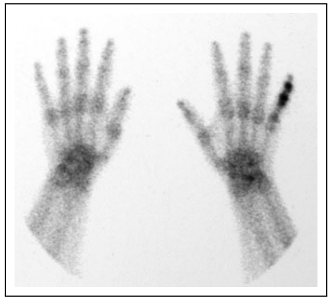
Network systems in sciences

Sociology: opinion dynamics, propagation of information, performance of teams

Ecology: ecosystems and foodwebs
Economics: input-output models
Medicine/Biology: compartmental systems

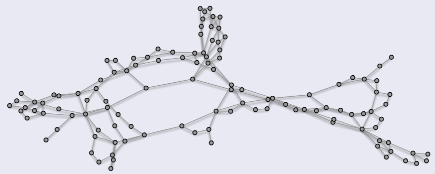






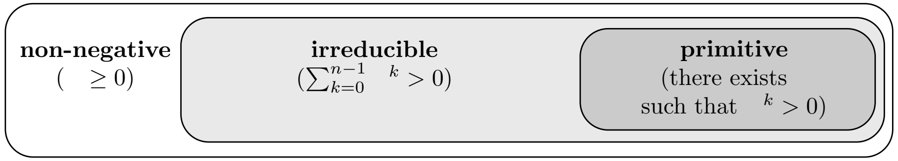
Linear network systems

$$x(k+1) = Ax(k) + b \qquad \text{or} \qquad \dot{x}(t) = Ax(t) + b$$



network structure $\iff$ function = asymptotic behavior

Perron-Frobenius theory



if A **non-negative**

1. eigenvalue $\lambda \geq |\mu|$ for all other eigenvalues μ
2. right and left eigenvectors $v_{\text{right}} \geq 0$ and $v_{\text{left}} \geq 0$

if A **irreducible**

3. $\lambda > 0$ and λ is simple
4. $v_{\text{right}} > 0$ and $v_{\text{left}} > 0$ are unique

if A **primitive**

5. $\lambda > |\mu|$ for all other eigenvalues μ
6. $\lim_{k\to\infty} A^k/\lambda^k = v_{\text{right}} v_{\text{left}}^T$, with normalization $v_{\text{right}}^T v_{\text{left}} = 1$

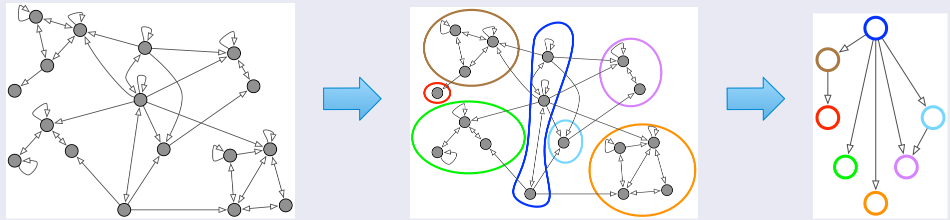
Algebraic graph theory

Powers of A ∼ paths in G:

$(A^k)_{ij} > 0$ ⇔ there exists directed path of length k from i to j in G

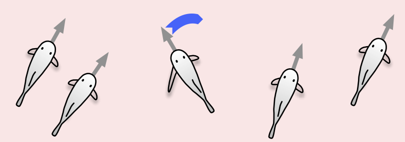
Primitivity of A ∼ paths in G:

A is primitive ($A \geq 0$ and $A^k > 0$) ⇔ G strongly connected and aperiodic (exists path between any two nodes) and (exists no k dividing each cycle length)



digraph → strongly connected components → **condensation**

Averaging systems



Swarming via averaging (French '56, DeGroot '74)

$x_i^+ :=$ average$(x_i, \{x_j, j$ is neighbor of $i\})$

⇓

$x(k+1) = Ax(k)$

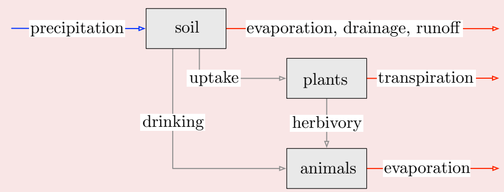
A influence matrix:
row-stochastic: non-negative and row-sums equal to 1

For general G with multiple condensed sinks
(assuming each condensed sink is aperiodic)

⇒ consensus at sinks
convex combinations elsewhere

consensus: $\lim_{k\to\infty} x(k) = (v_{\text{left}} \cdot x(0)) \mathbb{1}_n$
where v_{left} = left dominant eigenvector is social power

Network flow systems



Water flow model for a desert ecosystem (Noy-Meir '73)

$$\dot{q}_i = \sum_j (F_{j\to i} - F_{i\to j}) - F_{i\to 0} + u_i$$

⇓ $F_{i\to j} = f_{ij} q_i, \quad F = [f_{ij}]$

$$\dot{q} = \underbrace{\left(F^T - \text{diag}(F\mathbb{1}_n + f_0)\right)}_{=:C} q + u$$

C compartmental matrix:
quasi-positive (off-diag ≥ 0) and non-positive column sums ($f_0 \geq 0$)
analysis tools: *PF for quasi-positive, inverse positivity, algebraic graph*

system (= each condensed sink) is outflow-connected ⇔ C is Hurwitz

⇒ $\lim_{t\to\infty} q(t) = -C^{-1}u \geq 0$
$(-C^{-1}u)_i > 0 \iff i$th compartment is inflow-connected

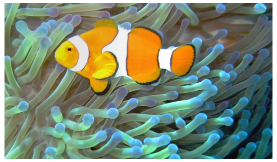
Nonlinear network systems

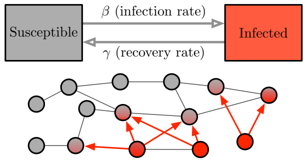
Rich variety of emerging behaviors

1. equilibria / limit cycles / extinction in populations dynamics
2. epidemic outbreaks in spreading processes
3. **synchrony and multi-stability in coupled oscillators**

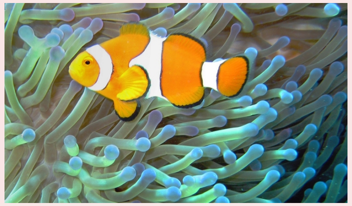
Rich variety of analysis tools

1. nonlinear stability theory
2. passivity and dissipativity
3. contractivity and monotonicity





Population systems in ecology



Mutualism clownfish / anemones (Takeuchi et al '78)

Lotka-Volterra: x_i = quantity/density

$$\frac{\dot{x}_i}{x_i} = b_i + \sum_j a_{ij} x_j$$

$$\dot{x} = \mathrm{diag}(x)(Ax + b)$$

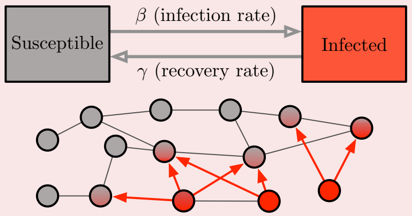
interaction matrix A:

$(+,+)$ mutualism, $(+,-)$ predation, $(-,-)$ competition
rich behavior: persistence, extinction, equilibria, periodic orbits, ...

1. **bounded resources:** A Hurwitz (e.g., irreducible and neg diag dom)
2. **logistic growth:** $b_i > 0$ and $a_{ii} < 0$
3. **mutualism:** $a_{ij} \geq 0$

⇒ exists unique steady state $-A^{-1}b > 0$
$\lim_{t\to\infty} x(t) = -A^{-1}b$ from all $x(0) > 0$

Network propagation in epidemiology



Network SIS model (Lajmanovich & Yorke '76)

Network SIS: (x_i = infected fraction)

$$\dot{x}_i = \beta \sum_j a_{ij}(1 - x_i)x_j - \gamma x_i$$

(rescaling)

$$\dot{x} = (I_n - \mathrm{diag}(x))Ax - x$$

contact matrix A: irreducible with dominant pair $(\lambda, v_{\mathrm{right}})$

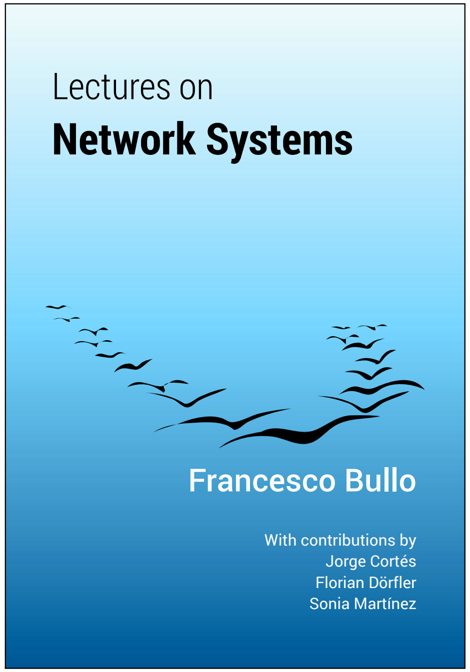
below the epidemic threshold: $\lambda < 1$

⇒ 0 is unique stable equilibrium
$v_{\mathrm{right}}^T x(t) \to 0$ monotonically & exponentially

above the epidemic threshold: $\lambda > 1$

⇒ 0 is unstable equilibrium
unique other equilibrium $x^* > 0$
$\lim_{t\to\infty} x(t) = x^*$

New text "Lectures on Network Systems"



Lectures on Network Systems, Francesco Bullo, Createspace, 1 edition, 2018, ISBN 978-1-986425-64-3

1. Self-Published and Print-on-Demand at:
   https://www.amazon.com/dp/1986425649
2. PDF Freely available at
   http://motion.me.ucsb.edu/book-lns:
   For students: free PDF for download
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3. incorporates lessons from 2 decades of research:
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4. now v1.2
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300 pages
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1. **Introduction to Network Systems**
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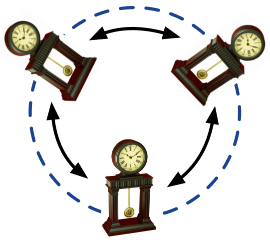
Today: Sync & Multi-Stability in Coupled Oscillators

Kuramoto model

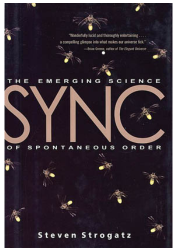
- n **oscillators** with angle $\theta_i \in \mathbb{S}^1$
- **non-identical** natural frequencies $\omega_i \in \mathbb{R}^1$
- **coupling** with strength $a_{ij} = a_{ji}$

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$



Motivation

1. Pendulum clocks & *"an odd kind of sympathy"*
   [Christiaan Huygens, Horologium Oscillatorium, 1673]
2. Local canonical model for weakly-coupled limit-cycle oscillators
   [Hoppensteadt et al. '97, Brown et al. '04]
3. *Simplest "network system on manifold" with rich phenomenology*

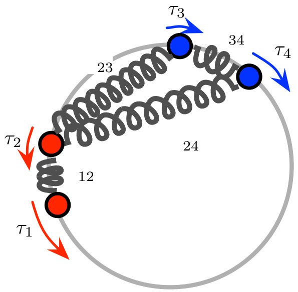


1. Countless sync phenomena in sciences/engineering
   scholar.google: *Winfree '67* 1.5K, *Kuramoto '75* 6.8K,
   surveys by Strogatz, Acebron, Arenas: 2K citations each

Applications in sciences: **biology:** pacemaker cells in the heart, circadian cells in the brain, coupled cortical neurons, Hodgkin-Huxley neurons, brain networks, yeast cells, flashing fireflies, chirping crickets, central pattern generators for animal locomotion, particle models mimicking animal flocking behavior, and fish schools
physics and chemistry: spin glass models,

Applications in engineering: deep brain stimulation, locking in solid-state circuit oscillators, planar vehicle coordination, carrier synchronization without phase-locked loops, semiconductor laser arrays, and microwave oscillator arrays
electric applications: structure-preserving and network-reduced power system models, and droop-controlled inverters in microgrids

Model #1: Spring network analog and applications



Coupled swing equations
Euler-Lagrange eq for spring network on ring:

$$m_i \ddot{\theta}_i + d_i \dot{\theta}_i = \tau_i - \sum_j k_{ij} \sin(\theta_i - \theta_j)$$

Kuramoto coupled oscillators

$$\dot{\theta}_i = \omega_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$$

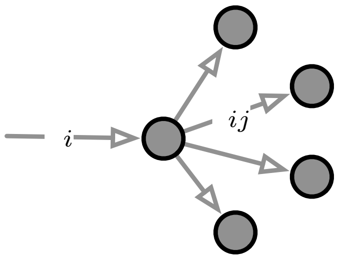
Kuramoto equilibrium equation

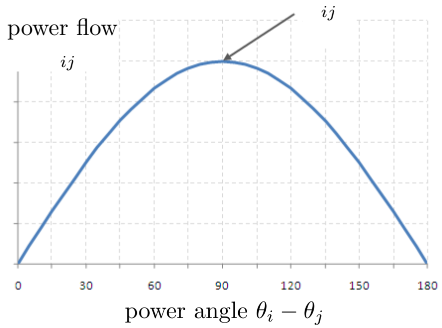
$$0 = \omega_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$$

Model #2: Active Power Flow Problem

AC, Kirchhoff and Ohm, quasi-sync, lossless lines, constant voltages.
supply/demand p_i, max power coeff a_{ij}, voltage phase θ_i

$$p_i = \sum_{j=1}^{n} f_{ij}, \qquad f_{ij} = a_{ij} \sin(\theta_i - \theta_j)$$





Given: network parameters & topology, load & generation profile,

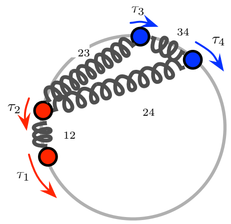
Flow / spring analogy

Active power flow problem

$$p_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

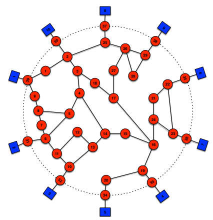
Spring network

- $p_i = \tau_i$: torque at i
- $a_{ij} = k_{ij}$: spring stiffness i, j
- $\sin(\theta_i - \theta_j)$: modulation



Power network

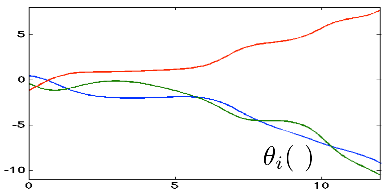
- p_i : injected power
- a_{ij} : max power flow i, j
- $\sin(\theta_i - \theta_j)$: modulation



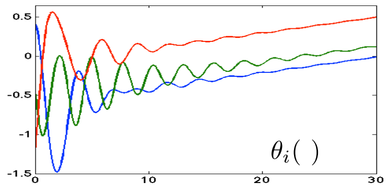
Phenomenon #1: Transition from incoherence to sync

Function = synchronization

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$



large $|\omega_i - \omega_j|$ & small coupling
$\Rightarrow$ incoherence = no sync



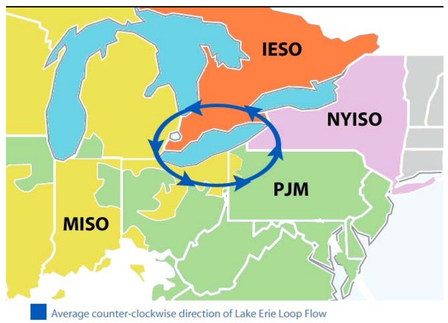
small $|\omega_i - \omega_j|$ & large coupling
$\Rightarrow$ coherence = frequency sync

- threshold: "heterogeneity" vs. "coupling"
- quantify: "heterogeneity" < "coupling"
- as function of network parameters

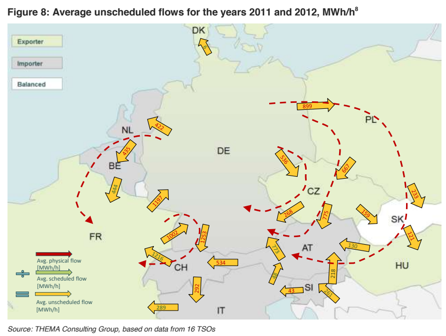
Phenomenon #2: Multiple power flows

Theoretical observation: multiple solutions exist

Practical observations:
sometimes undesirable power flows around loops
sometimes sizable difference between predicted and actual power flows



New York Independent System Operator, Lake Erie Loop Flow Mitigation, Technical Report, 2008



THEMA Consulting Group, Loop-flows - Final advice, Technical Report prepared for the European Commission, 2013

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Primer on algebraic graph theory

Weighted undirected graph with n nodes and m edges:

Incidence matrix: $n \times m$ matrix B s.t. $(B^\top p_{\text{active}})_{(ij)} = p_i - p_j$

Weight matrix: $m \times m$ diagonal matrix $\mathcal{A}$

Laplacian stiffness: $L = B\mathcal{A}B^\top \geq 0$

Linearization of Kuramoto equilibrium equation:

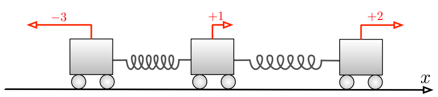
$$p_{\text{active}} = B\mathcal{A}\sin(B^\top\theta) \quad \Longrightarrow \quad p_{\text{active}} \approx B\mathcal{A}(B^\top\theta) = L\,\theta$$

Algebraic connectivity:

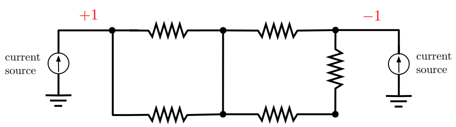
$$\lambda_2(L) = \text{second smallest eig of } L$$
$$= \text{notion of connectivity and coupling}$$

Laplacian linear balance equation

Linear spring and resistive networks



(a) spring network



(b) resistive circuit

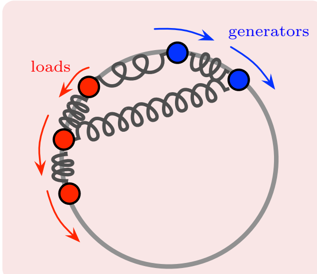
$L_{\text{stiffness}}\, x = f_{\text{load}}$ and $L_{\text{conductance}}\, v = c_{\text{injected}}$

Laplacian linear balance equation: $p_{\text{active}} = L\,\theta$

if $\sum_i p_i = 0$ in $p_{\text{active}} = L\,\theta$, then equilibrium exists : $\theta = L^\dagger p_{\text{active}}$

pairwise displacements : $B^\top\theta = B^\top L^\dagger p_{\text{active}}$

From Old to New Tests



Given balanced p_{active}, do angles exist?

$$p_{\text{active}} = B\mathcal{A}\sin(B^\top\theta)$$

synchronization arises if
heterogeneity < coupling
power transmission < connectivity strength

Old Tests: Equilibrium angles (neighbors within $\pi/2$ arc) exist if

$\lVert B^\top p_{\text{active}}\rVert_2$	$< \lambda_2(L)$	for unweighted graphs	(Old 2-norm T)
$\lVert B^\top L^\dagger p_{\text{active}}\rVert_\infty$	< 1	for trees, complete	(Old ∞-norm T)

From Old to New Tests

Old Tests: Equilibrium angles (neighbors within $\pi/2$ arc) exist if

$\|B^\top p_{\text{active}}\|_2 < \lambda_2(L)$	for unweighted graphs	(Old 2-norm T)
$\|B^\top L^\dagger p_{\text{active}}\|_\infty < 1$	for trees, complete	(Old ∞-norm T)

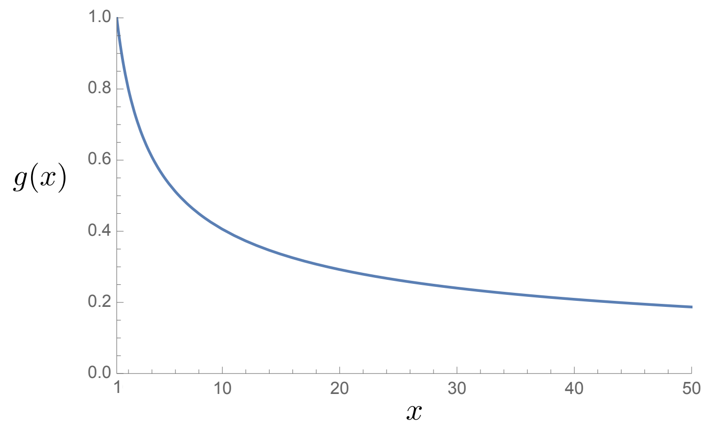
New Tests: Equilibrium angles (neighbors within $\pi/2$ arc) exist if

$\|B^\top L^\dagger p_{\text{active}}\|_2 < 1$	for unweighted graphs	(New 2-norm T)
$\|B^\top L^\dagger p_{\text{active}}\|_\infty < g(\|\mathcal{P}\|_\infty)$	for all graphs	(New ∞-norm T)

where g is monotonically decreasing

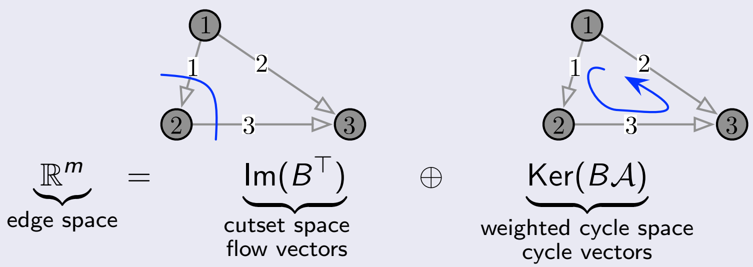
$$g : [1,\infty) \to [0,1]$$

$$x \mapsto \frac{y(x) + \sin(y(x))}{2} - x\,\frac{y(x) - \sin(y(x))}{2}\Big|_{y(x) = \arccos(\frac{x-1}{x+1})}$$



and where $\mathcal{P}$ is a projection matrix

$$\mathcal{P} = B^\top L^\dagger B\mathcal{A} \quad = \text{oblique projection onto } \mathrm{Im}(B^\top) \text{ parallel to } \mathrm{Ker}(B\mathcal{A})$$



1. if G unweighted, then $\mathcal{P}$ is orthogonal and $\|\mathcal{P}\|_2 = 1$
2. if G acyclic, then $\mathcal{P} = I_m$ and $\|\mathcal{P}\|_p = 1$
3. if G uniform complete or ring, then $\|\mathcal{P}\|_\infty = 2(n-1)/n \leq 2$

New Tests: Equilibrium angles (neighbors within $\pi/2$ arc) exist if

$\|B^\top L^\dagger p_{\text{active}}\|_2 < 1$	for unweighted graphs	(New 2-norm T)
$\|B^\top L^\dagger p_{\text{active}}\|_\infty < g(\|\mathcal{P}\|_\infty)$	for all graphs	(New ∞-norm T)

Unifying theorem with a family of tests

Equilibrium angles (neighbors within γ arc) exist if, in some p-norm,

$$\|B^\top L^\dagger p_{\text{active}}\|_p \leq \gamma \alpha_p(\gamma) \qquad \text{for all graphs} \qquad (\text{New } \alpha_p \text{ T})$$

where nonconvex optimization problem:

$$\alpha_p(\gamma) := \text{ min amplification factor of } \quad \mathcal{P}\,\mathrm{diag}[\mathrm{sinc}(x)]$$

Comparison of sufficient and approximate sync tests

Any test predicts max transmittable power (before bifurcation).
Compare with numerically computed.

Test Case	ratio of test prediction to numerical computation			
	old 2-norm	new ∞-norm	$g(\|\mathcal{P}\|_\infty) \approx 1$ approximate	α_∞ test *fmincon*
IEEE 9	16.54 %	73.74 %	92.13 %	85.06 %†
IEEE 14	8.33 %	59.42 %	83.09 %	81.32 %†
IEEE RTS 24	3.86 %	53.44 %	89.48 %	89.48 %†
IEEE 30	2.70 %	55.70 %	85.54 %	85.54 %†
IEEE 118	0.29 %	43.70 %	85.95 %	—*
IEEE 300	0.20 %	40.33 %	99.80 %	—*
Polish 2383	0.11 %	29.08 %	82.85 %	—*

† *fmincon* with 100 randomized initial conditions
* *fmincon* does not converge

Summary: Kuramoto equilibrium and active power flow

Given topology (incidence B), admittances (Laplacian L), injections p_{active},

$$p_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

Equilibrium angles exist if, in some p-norm,

$$\|B^\top L^\dagger p_{\text{active}}\|_p \leq \gamma \alpha_p(\gamma) \qquad \text{for all graphs} \qquad (\text{New } \alpha_p \text{ T})$$

For $p = \infty$, after bounding,

$$\|B^\top L^\dagger p_{\text{active}}\|_\infty \leq g(\|\mathcal{P}\|_\infty) \qquad (\text{New } \infty\text{-norm T})$$

Q1: $\exists$ a **stable operating point** (with pairwise angles $\leq \gamma$)?
Q2: what is the **network capacity** to transmit active power?
Q3: how to quantify **robustness** as distance from loss of feasibility?

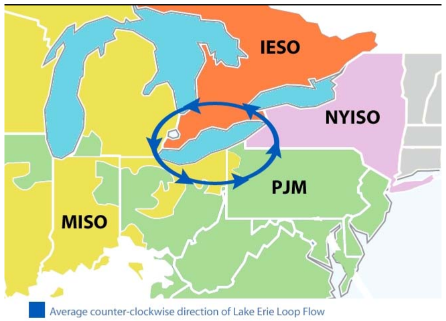
Outline

1. Introduction to Network Systems
   F. Bullo. *Lectures on Network Systems.*
   CreateSpace, 1 edition, 2018.
   With contributions by J. Cortés, F. Dörfler, and S. Martínez.
   URL: `http://motion.me.ucsb.edu/book-lns`
2. Synchronization (existence)
   S. Jafarpour and F. Bullo. Synchronization of Kuramoto oscillators via cutset projections.
   IEEE Transactions on Automatic Control, 2018.
   `doi:10.1109/TAC.2018.2876786`
3. Multi-Stability (lack of uniqueness)
   S. Jafarpour, E. Y. Huang, K. D. Smith, and F. Bullo. Multistable synchronous power flows: From geometry to analysis and computation.
   January 2019.
   URL: `https://arxiv.org/pdf/1901.11189.pdf`

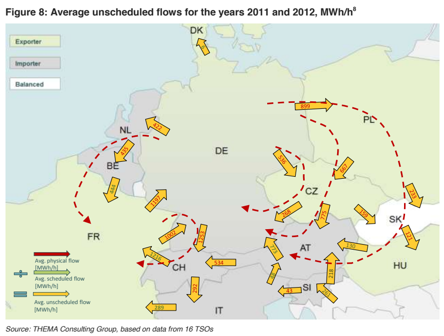
Phenomenon #2: Multiple power flows

Theoretical observation: multiple solutions exist

Practical observations:
- sometimes undesirable power flows around loops
- sometimes sizable difference between predicted and actual power flows



New York Independent System Operator, **Lake Erie Loop Flow Mitigation**, Technical Report, 2008



THEMA Consulting Group, **Loop-flows - Final advice**, Technical Report prepared for the European Commission, 2013

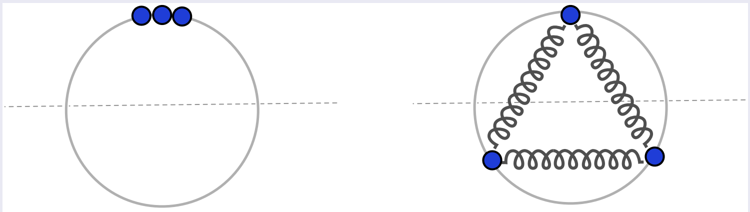
Lack of uniqueness and winding solutions

Given topology (incidence B), admittances (Laplacian L), injections p_{active},

$$p_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

1. is solution unique?
2. how to localize/classify solutions?

triangle graph, homogeneous weights ($a_{ij} = 1$), $p_{\text{active}} = 0$



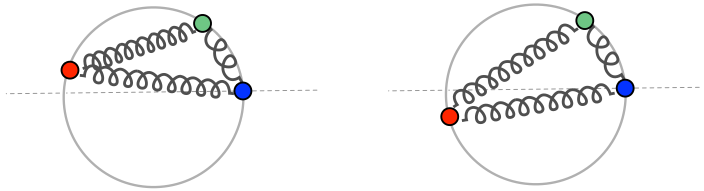
phase sync   splay state

Winding number of n angles

Given undirected graph with a cycle $\sigma = (1, \dots, n_\sigma)$ and orientation

1. **winding number of $\theta \in \mathbb{T}^n$ along σ** is:

$$w_\sigma(\theta) = \frac{1}{2\pi} \sum_{i=1}^{n_\sigma} d_{\text{cc}}(\theta_i, \theta_{i+1})$$



$(\theta) = 0$ $(\theta) = \pm 1$

2. given basis $\sigma_1, \dots, \sigma_r$ for cycles, **winding vector of θ** is

$$w(\theta) = (w_{\sigma_1}(\theta), \dots, w_{\sigma_r}(\theta))$$

"Kirchhoff Angle Law" and partition of the n-torus

Theorem: Kirchhoff angle law on $\mathbb{T}^n$

$$w_\sigma(\theta) = 0, \pm 1, \dots, \pm \lfloor n_\sigma / 2 \rfloor$$

$\Longrightarrow$ $w(\theta)$ is piecewise constant

$\Longrightarrow$ $w(\theta)$ takes value in a finite set

Theorem: Winding partition

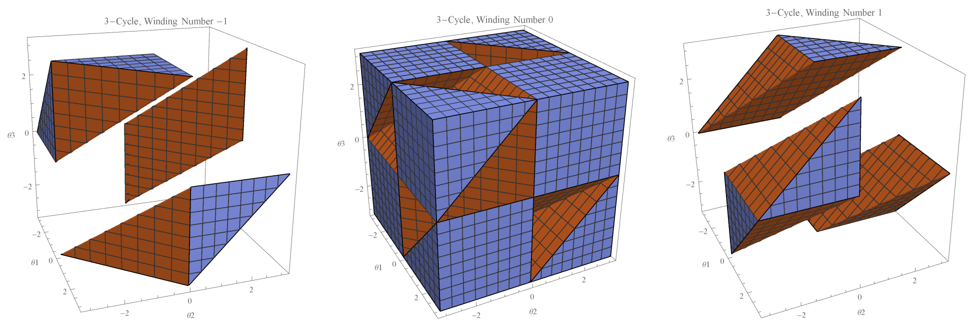
For each possible winding vector u, define

$$\text{WindingCell}(u) := \{\theta \in \mathbb{T}^n \mid w(\theta) = u\}$$

Then

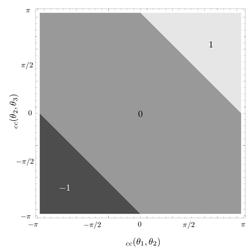
$$\mathbb{T}^n = \cup_u \text{WindingCell}(u)$$

Winding partition of triangle graph



$w = -1$ $w = 0$ $w = +1$

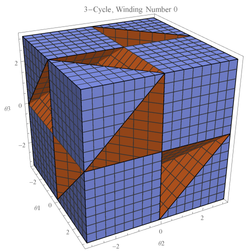
- each winding cell is connected
- each winding cell is invariant under rotation
- bijection:
  reduced winding cell $\longleftrightarrow$ open convex polytope



The Kuramoto model and the winding partition

Given topology (incidence B), admittances (Laplacian L), injections p_{active},

$$\dot{\theta}_i = p_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$$



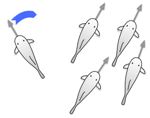
Theorem: At-most-uniqueness and extensions

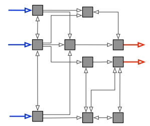
1. each WindingCell has at-most-unique equilibrium with $\Delta\theta < \pi/2$
2. equilibrium loop flow increases monotonically wrt winding number
3. existence + uniqueness in WindingCell(u) with $\Delta\theta < \pi/2$ if

$$\|B^\top L^\dagger p_{\text{active}} + Cu\|_\infty \leq g(\|\mathcal{P}\|_\infty), \text{ or} \qquad \text{(Static T)}$$

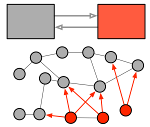
$$\exists \text{ a trajectory inside WindingCell}(u) \text{ with } \Delta\theta < \pi/2 \qquad \text{(Dynamic T)}$$

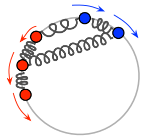
Summary and Future Work

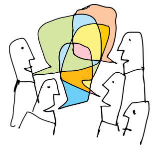

averaging


compartmental flows


mutualism


virus spread


coupled oscillators


social systems

Contributions

1. an emergent theory of network systems
2. trade-off between coupling strength and oscillator heterogeneity
3. algebraic graph theory of the torus

Future research

1. close the gap between sufficient and necessary conditions
2. more realistic power flow equations
3. applications to other dynamic flow networks
4. **outreach/collaboration opportunities for our community with** sociologists, biologists, economists, physicists ...