Distributed Control and Coordination Algorithms

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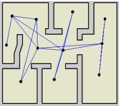
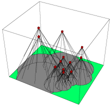
Cooperative multi-agent systems

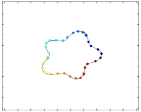
What kind of systems?
Groups of agents with control, sensing, communication and computing

Each individual

- **senses** its immediate environment
- **communicates** with others
- **processes** information gathered
- **takes local action** in response

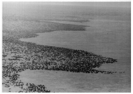




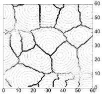


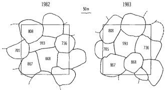
Self-organized behaviors in biological groups

motion patterns in 1, 2 and 3 dimensions



territory partitioning in fish, ants and sparrows





Decision making in animals

Able to

- forage over a given region
- assume specified pattern
- rendezvous at a common point
- jointly initiate motion/change direction in a synchronized way

Species achieve synchronized behavior

- with limited sensing/communication between individuals
- without apparently following group leader

References

- L. Conradt and T. J. Roper. Group decision-making in animals. *Nature*, 421(6919):155–158, 2003
- I. D. Couzin, J. Krause, N. R. Franks, and S. A. Levin. Effective leadership and decision-making in animal groups on the move. *Nature*, 433(7025):513–516, 2005

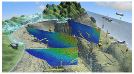
Engineered multi-agent systems

Embedded robotic systems and sensor networks for

- high-stress, rapid deployment — e.g., disaster recovery networks
- distributed environmental monitoring — e.g., portable chemical and biological sensor arrays detecting toxic pollutants
- autonomous sampling for biological applications — e.g., monitoring of species in risk, validation of climate and oceanographic models
- science imaging — e.g., multispacecraft distributed interferometers flying in formation to enable imaging at microarcsecond resolution


Sandia National Labs


UCSD Scripps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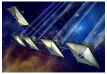

NASA

Research challenges

What useful engineering tasks can be performed
with limited-sensing/communication agents?

Dynamics	simple interactions give rise to rich emerging behavior
Feedback	rather than open-loop computation for known/static setup
Information flow	who knows what, when, why, how, dynamically changing
Reliability/performance	robust, efficient, predictable behavior

How to coordinate individual agents into coherent whole?

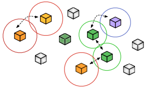
Objective: *systematic methodologies to design and analyze cooperative strategies to control multi-agent systems*

Research objectives

Design of provably correct coordination algorithms for basic tasks

Formal model to rigorously formalize, analyze, and compare coordination algorithms

Mathematical tools to study convergence, stability, and robustness of coordination algorithms



Coordination tasks

exploration, map building, search and rescue,
surveillance, odor localization, monitoring, distributed sensing

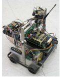
Technical approach

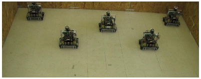
Optimization Methods
- resource allocation
- geometric optimization
- load balancing

Geometry & Analysis
- computational structures
- differential geometry
- nonsmooth analysis

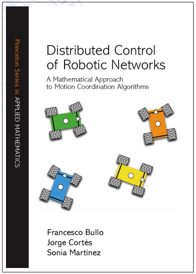
Control & Robotics
- algorithm design
- cooperative control
- stability theory

Distributed Algorithms
- adhoc networks
- decentralized vs centralized
- emerging behaviors





Text: Distributed Control of Robotic Networks



1. intro to distributed algorithms (graph theory, synchronous networks, and averaging algos)
2. geometric models and geometric optimization problems
3. model for robotic, relative sensing networks, and complexity
4. algorithms for rendezvous, deployment, boundary estimation

F. Bullo, J. Cortés, and S. Martínez. *Distributed Control of Robotic Networks*. Applied Mathematics Series. Princeton University Press, 2009. Available at http://www.coordinationbook.info

Outline

Models for multi-agent networks

References

1. I. Suzuki and M. Yamashita. Distributed anonymous mobile robots: Formation of geometric patterns. *SIAM Journal on Computing*, 28(4):1347–1363, 1999
2. N. A. Lynch. *Distributed Algorithms*. Morgan Kaufmann, 1997
3. D. P. Bertsekas and J. N. Tsitsiklis. *Parallel and Distributed Computation: Numerical Methods*. Athena Scientific, 1997
4. S. Martínez, F. Bullo, J. Cortés, and E. Frazzoli. On synchronous robotic networks – Part I: Models, tasks and complexity. *IEEE Transactions on Automatic Control*, 52(12):2199–2213, 2007

Objective

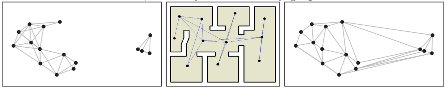
1. meaningful + tractable model
2. feasible operations and their cost
3. control/communication tradeoffs

Robotic network

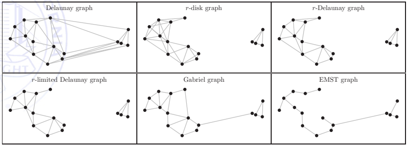
A **uniform/anonymous robotic network** $\mathcal{S}$ is

1. $I = \{1, \dots, N\}$; **set of unique identifiers (UIDs)**
2. $\mathcal{A} = \{A^{[i]}\}_{i \in I}$, with $A^{[i]} = (X, U, f)$ is a **set of physical agents**
3. **interaction graph**

Disk, visibility and Delauney graphs



Communication models for robotic networks



Relevant graphs

1. fixed, directed, balanced
2. switching
3. geometric or state-dependent
4. random, random geometric

Message model

1. message
2. packet/bits
3. absolute or relative positions
4. packet losses

Prototypical examples

Locally-connected first-order robots in $\mathbb{R}^d$ $\quad \mathcal{S}_{\text{disk}}$

- n points $x^{[1]}, \dots, x^{[n]}$ in $\mathbb{R}^d$, $d \geq 1$
- obeying $\dot{x}^{[i]}(t) = u^{[i]}(t)$, with $u^{[i]} \in [-u_{\max}, u_{\max}]$
- identical robots of the form

$$(\mathbb{R}^d, [-u_{\max}, u_{\max}]^d, \mathbb{R}^d, (\mathbf{0}, \boldsymbol{e}_1, \dots, \boldsymbol{e}_d))$$

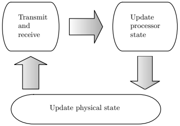
- each robot communicates to other robots within r

Variations

1. $\mathcal{S}_{\text{D}}$ same dynamics, but Delaunay graph
2. $\mathcal{S}_{\text{LD}}$: same dynamics, but r-limited Delaunay graph
3. $\mathcal{S}_{\text{vehicles}}$: same graph, but nonholonomic dynamics

Synchronous control and communication

1. **communication schedule** $\quad \mathbb{T} = \{t_\ell\}_{\ell \in \mathbb{N}_0} \subset \mathbb{R}_{\geq 0}$
2. **communication alphabet** $\quad L$ including the `null` message
3. set of values for **logic variables** $\quad W$
4. **message-generation function** $\quad \text{msg}\colon \mathbb{T} \times X \times W \times I \to L$
5. **state-transition functions** $\quad \text{stf}\colon \mathbb{T} \times W \times L^N \to W$
6. **control function** $\quad \text{ctrl}\colon \mathbb{R}_{\geq 0} \times X \times W \times L^N \to U$



Task and complexity

- **Coordination task** is $(\mathcal{W}, \mathcal{T})$ where $\mathcal{T}\colon X^N \times \mathcal{W}^N \to \{\texttt{true}, \texttt{false}\}$
  1. **Logic-based:** achieve consensus, synchronize, form a team
  2. **Motion:** deploy, gather, flock, reach pattern
  3. **Sensor-based:** search, estimate, identify, track, map
- For $\{\mathcal{S}, \mathcal{T}, \mathcal{CC}\}$, define costs/complexity:
  control effort, communication packets, computational cost
- **Time complexity to achieve $\mathcal{T}$ with $\mathcal{CC}$**

$$\mathsf{TC}(\mathcal{T}, \mathcal{CC}, x_0, w_0) = \inf \{\ell \mid \mathcal{T}(x(t_k), w(t_k)) = \texttt{true}, \text{ for all } k \geq \ell\}$$

$$\mathsf{TC}(\mathcal{T}, \mathcal{CC}) = \sup \{ \mathsf{TC}(\mathcal{T}, \mathcal{CC}, x_0, w_0) \mid (x_0, w_0) \in X^N \times \mathcal{W}^N \}$$

$$\mathsf{TC}(\mathcal{T}) = \inf \{ \mathsf{TC}(\mathcal{T}, \mathcal{CC}) \mid \mathcal{CC} \text{ achieves } \mathcal{T} \}$$

Outline

1. Models for multi-agent networks
2. Rendezvous and connectivity maintenance
   - Maintaining connectivity
   - Circumcenter algorithms
   - Correctness analysis
   - Time complexity analysis
3. Deployment
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4. Summary and conclusions

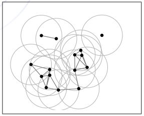
Rendezvous and connectivity maintenance

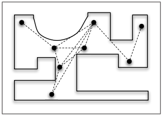
References

1. H. Ando, Y. Oasa, I. Suzuki, and M. Yamashita. Distributed memoryless point convergence algorithm for mobile robots with limited visibility. *IEEE Transactions on Robotics and Automation*, 15(5):818–828, 1999
2. Z. Lin, M. Broucke, and B. Francis. Local control strategies for groups of mobile autonomous agents. *IEEE Transactions on Automatic Control*, 49(4):622–629, 2004
3. P. Flocchini, G. Prencipe, N. Santoro, and P. Widmayer. Gathering of asynchronous oblivious robots with limited visibility. *Theoretical Computer Science*, 337(1-3):147–168, 2005
4. J. Lin, A. S. Morse, and B. D. O. Anderson. The multi-agent rendezvous problem. Part 1: The synchronous case. *SIAM Journal on Control and Optimization*, 46(6):2096–2119, 2007
5. J. Cortés, S. Martínez, and F. Bullo. Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions. *IEEE Transactions on Automatic Control*, 51(8):1289–1298, 2006
6. S. Martínez, F. Bullo, J. Cortés, and E. Frazzoli. On synchronous robotic networks – Part II: Time complexity of rendezvous and deployment algorithms. In *IEEE Conf. on Decision and Control and European Control Conference*, pages 8313–8318, Seville, Spain, December 2005
7. A. Ganguli, J. Cortés, and F. Bullo. Multirobot rendezvous with visibility sensors in nonconvex environments. *IEEE Transactions on Robotics*, 25(2):340–352, 2009

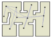
Rendezvous coordination task

Objective:
achieve multi-robot rendezvous; i.e. arrive at the same location of space, while maintaining connectivity

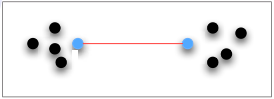




r-disk connectivity 

visibility connectivity 

We have to be careful...



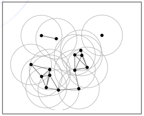
Blindly "getting closer" to neighboring agents might break overall connectivity

Outline

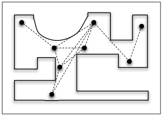
Constraint sets for connectivity

Design constraint sets with key properties

- Constraints are flexible enough so that network does not get stuck
- Constraints change continuously with agents' position



r-disk connectivity

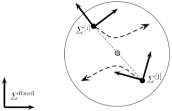


visibility connectivity

Enforcing range-limited links – pairwise

Definition (Pairwise connectivity maintenance problem)

Given two neighbors in $\mathcal{G}_{\text{disk}}(r)$, find a rich set of control inputs for both agents with the property that, after moving, both agents are again within distance r

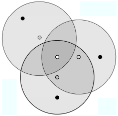


If $\mathsf{dist}(p^{[i]}(\ell), p^{[j]}(\ell)) \leq r$, and remain in ball of radius $r/2$ (connectivity set), then $\mathsf{dist}(p^{[i]}(\ell+1), p^{[j]}(\ell+1)) \leq r$

Enforcing range-limited links – w/ all neighbors

Definition (Connectivity constraint set)

Given a group of agents at positions $P = \{p^{[1]}, \ldots, p^{[n]}\} \subset \mathbb{R}^d$
The *connectivity constraint set* of agent i with respect to P is *intersection* of pairwise connectivity constraint set

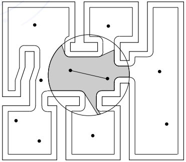


Enforcing range-limited line-of-sight links – pairwise

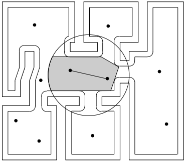
Given nonconvex $Q \subset \mathbb{R}^2$, contraction is $Q_\delta = \{q \in Q \mid \mathsf{dist}(q, \partial Q) \geq \delta\}$

Pairwise connectivity maintenance problem:

Given two neighbors in $\mathcal{G}_{\text{vis-disk},Q_\delta}$, find a rich set of control inputs for both agents with the property that, after moving, both agents are again within distance r and visible to each other in Q_δ



for each pair of visible robots

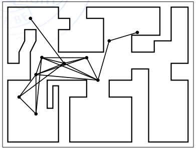


visibility pairwise constraint set

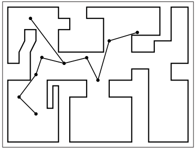
Topology matters

Connectivity constraint procedure over sparser graphs $\implies$ fewer constraints:

1. select a graph that has same connected components
2. select a graph whose edges can be computed in a distributed way



visibility graph



locally-cliqueless visibility graph

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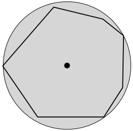
Circumcenter control and communication law

circumcenter $\mathsf{CC}(W)$ of bounded set W is center of closed ball of minimum radius containing W

circumradius $\mathsf{CR}(W)$ is radius of this ball

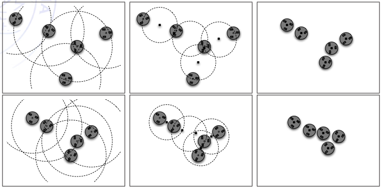


[Informal description:]

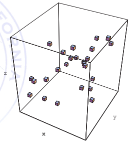
At each communication round each agent:
(i) transmits its position and receives its neighbors' positions
(ii) computes circumcenter of point set comprised of its neighbors and of itself
(iii) moves toward this circumcenter point while remaining inside constraint set

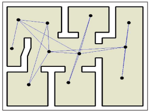
Circumcenter control and communication law

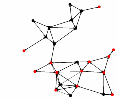
Illustration of the algorithm execution

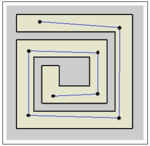


Simulations









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Formal algorithm description

`Robotic Network:` $\mathcal{S}_{\rm disk}$ with a discrete-time motion model,
with absolute sensing of own position, and
with communication range r, in $\mathbb{R}^d$

`Distributed Algorithm:` circumcenter

`Alphabet:` $L = \mathbb{R}^d \cup \{\texttt{null}\}$

`function` $\mathrm{msg}(p, i)$

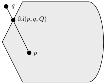
return p

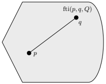
`function` $\mathrm{ctrl}(p, y)$

$p_{\rm goal} := \mathsf{CC}(\{p\} \cup \{p_{\rm rcvd} \mid \text{for all non-null } p_{\rm rcvd} \in y\})$

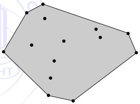
$\mathcal{X} := \mathcal{X}_{\rm disk}(p, \{p_{\rm rcvd} \mid \text{for all non-null } p_{\rm rcvd} \in y\})$

return $\mathsf{fti}(p, p_{\rm goal}, \mathcal{X}) - p$

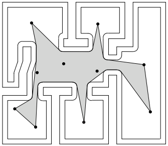




Some good news: Lyapunov functions



convex hull



relative convex hull

Lyapunov function: diameter or perimeter of convex hull

Let S be a set of points in $\mathbb{R}^d$

1. $\mathsf{CC}(S)$ belongs to $\mathsf{co}(S) \setminus \mathsf{Ve}(\mathsf{co}(S))$
2. pick $p \in S \setminus \mathsf{CC}(S)$ and $r \geq \max_{q \in S} \|p - q\|$. Then, for all $q \in S$ the open segment $(p, \mathsf{CC}(S))$ has nonempty intersection with $B\left(\frac{p+q}{2}, \frac{r}{2}\right)$

Convergence thm #1: standard version

- map $T\colon X \to X$
- consider the sequence $\{x_\ell\}_{\ell \geq 0} \subset X$ with

$$x_{\ell+1} = T(x_\ell)$$

Assume:

1. $W \subset X$ compact and positively invariant for T
2. $U\colon W \to \mathbb{R}$ non-increasing along T
3. U and T are continuous on W

If $x_0 \in W$, then

$$x_\ell \to \{w \in W \mid U(T(w)) = U(w)\}$$

(more precisely, largest invariant set thereof, intersected with level set)

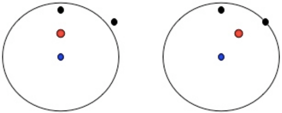
Some bad news

Circumcenter algorithms are nonlinear discrete-time dynamical systems

$$x_{\ell+1} = f(x_\ell)$$

To analyze convergence, we need at least f continuous – to use classic Lyapunov/LaSalle results

But circumcenter algorithms are discontinuous because of changes in interaction topology



Alternative idea

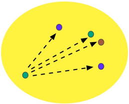
1. Fixed undirected graph G, define **fixed-topology circumcenter algorithm**

   $$f_G : (\mathbb{R}^d)^n \to (\mathbb{R}^d)^n, \quad f_{G,i}(p_1, \dots, p_n) = \mathsf{fti}(p, p_{\text{goal}}, \mathcal{X}) - p$$

   Now, there are no topological changes in f_G, hence f_G is continuous

2. Define set-valued map $T_{\mathsf{CC}} : (\mathbb{R}^d)^n \rightrightarrows (\mathbb{R}^d)^n$

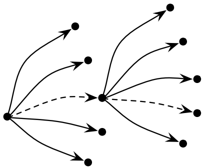
   $$T_{\mathsf{CC}}(p_1, \dots, p_n) = \{f_G(p_1, \dots, p_n) \mid G \text{ connected}\}$$



Convergence thm: nondeterminism

- set-valued $T : X \rightrightarrows X$ with $T(x) = \{T_i(x)\}_{i \in I}$ for finite I
- consider sequences $\{x_\ell\}_{\ell \geq 0} \subset X$ with

$$x_{\ell+1} \in T(x_\ell)$$



Convergence thm #2: arbitrary switches

- finite collection of maps $T_i \colon X \to X$ for $i \in I$
- consider a sequence $\{x_\ell\}_{\ell \geq 0} \subset X$ with

$$x_{\ell+1} = T_{i(\ell)}(x_\ell)$$

Assume:

1. $W \subset X$ compact and positively invariant for each T_i
2. $U \colon W \to \mathbb{R}$ non-increasing along each T_i
3. U and T_i are continuous on W

If $x_0 \in W$, then

$$x_\ell \to \{w \in W \mid U(T_i(w)) = U(w) \text{ for some } i\}$$

(more precisely, largest invariant set thereof, intersected with level set)

Correctness via LaSalle Invariance Principle

1. evolution starting from P_0 is contained in $\mathsf{co}(P_0)$
2. T_{CC} is finite collection of continuous maps
   each map is circumcenter algorithm at fixed connected topology
3. define U = diameter of convex hull = maximum pairwise distance
4. U is non-decreasing along each of the maps T_{CC}

Application of convergence thm: trajectories starting at P_0 converge to

$$\{P \in \mathsf{co}(P_0) \mid \exists P' \in T_{\mathsf{CC}}(P) \text{ such that } \mathsf{diam}(P') = \mathsf{diam}(P)\}$$

Additionally,

1. V is strictly decreasing unless all robots are coincident
2. all robots converge to a stationary point, again because $\mathsf{co}(P_0)$ is invariant

Correctness

Theorem (Correctness of the circumcenter laws)

For $d \in \mathbb{N}$, $r \in \mathbb{R}_{>0}$ and $\epsilon \in \mathbb{R}_{>0}$, the following statements hold:

1. *on $\mathcal{S}_{\text{disk}}$, the law $\mathcal{CC}_{\text{circumcenter}}$ (with control magnitude bounds and relaxed $\mathcal{G}$-connectivity constraints) achieves $\mathcal{T}_{\text{rendezvous}}$;*
2. *on $\mathcal{S}_{\text{LD}}$, the law $\mathcal{CC}_{\text{circumcenter}}$ achieves $\mathcal{T}_{\epsilon\text{-rendezvous}}$*

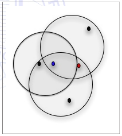
Furthermore,

1. *if any two agents belong to the same connected component at $\ell \in \mathbb{N}_0$, then they continue to belong to the same connected component subsequently; and*
2. *for each evolution, there exists $P^* = (p_1^*, \dots, p_n^*) \in (\mathbb{R}^d)^n$ such that:*
   1. *the evolution asymptotically approaches P^*, and*
   2. *for each $i, j \in \{1, \dots, n\}$, either $p_i^* = p_j^*$, or $\|p_i^* - p_j^*\|_2 > r$*

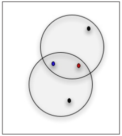
Similar result for visibility networks in non-convex environments

Robustness of circumcenter algorithms

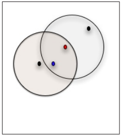
Push whole idea further!, e.g., for robustness against link failures



topology G_1



topology G_2



topology G_3

Look at **evolution under link failures** as outcome of nondeterministic evolution under multiple interaction topologies

$$P \longrightarrow \{\text{evolution under } G_1, \text{evolution under } G_2, \text{evolution under } G_3\}$$

Rendezvous

Corollary (Circumcenter algorithm over $\mathcal{G}_{\text{disk}}(r)$ on $\mathbb{R}^d$)

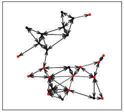
For $\{P_m\}_{m\in\mathbb{N}_0}$ synchronous execution with link failures such that union of any $\ell \in \mathbb{N}$ consecutive graphs in execution has globally reachable node

Then, there exists $(p^*, \dots, p^*) \in \mathsf{diag}((\mathbb{R}^d)^n)$ such that

$$P_m \to (p^*, \dots, p^*) \quad as \quad m \to +\infty$$

Proof uses

$$T_{\mathsf{CC},\ell}(P) = \{f_{\mathcal{G}_\ell} \circ \dots \circ f_{\mathcal{G}_1}(P) \mid \cup_{s=1}^{\ell} \mathcal{G}_i \text{ has globally reachable node}\}$$



Outline

1. Models for multi-agent networks
2. Rendezvous and connectivity maintenance
   - Maintaining connectivity
   - Circumcenter algorithms
   - Correctness analysis
   - Time complexity analysis
3. Deployment
   - Multi-center functions
   - Geometric-center laws
   - Peer-to-peer laws
   - Laws for disk-covering and sphere-packing
4. Summary and conclusions

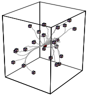
Correctness – Time complexity

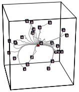
Theorem (Time complexity of circumcenter laws)

For $r \in \mathbb{R}_{>0}$ and $\epsilon \in]0,1[$, the following statements hold:

1. *on the network $\mathcal{S}_{\text{disk}}$, evolving on the real line $\mathbb{R}$ (i.e., with $d = 1$), $\mathsf{TC}(\mathcal{T}_{\text{rendezvous}}, \mathcal{CC}_{\text{circumcenter}}) \in \Theta(n)$;*
2. *on the network $\mathcal{S}_{\text{LD}}$, evolving on the real line $\mathbb{R}$ (i.e., with $d = 1$), $\mathsf{TC}(\mathcal{T}_{(r\epsilon)\text{-rendezvous}}, \mathcal{CC}_{\text{circumcenter}}) \in \Theta(n^2 \log(n\epsilon^{-1}))$; and*

Similar results for visibility networks





Time complexity proof techniques

For $N \geq 2$ and $a, b, c \in \mathbb{R}$, define the $N \times N$ Toeplitz matrices

$$\mathrm{Trid}_n(a,b,c) = \begin{bmatrix} b & c & 0 & \dots & 0 \\ a & b & c & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & a & b & c \\ 0 & \dots & 0 & a & b \end{bmatrix}$$

$$\mathrm{Circ}_n(a,b,c) = \mathrm{Trid}_n(a,b,c) + \begin{bmatrix} 0 & \dots & \dots & 0 & a \\ 0 & \dots & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ c & 0 & \dots & 0 & 0 \end{bmatrix}$$

To be studied for interesting a, b, c:
e.g., as stochastic matrices whose 2nd eigenvalue converges to 1 as $n \to +\infty$

Eigenvalues of tridiagonal Toeplitz and circulant mat.s

For $n \geq 2$ and $a, b, c \in \mathbb{R}$, the following statements hold:

1. for $ac \neq 0$, the eigenvalues and eigenvectors of $\mathrm{Trid}_n(a,b,c)$ are, for $i \in \{1,\dots,n\}$,

$$b + 2c\sqrt{\frac{a}{c}} \cos\left(\frac{i\pi}{n+1}\right), \text{ and}$$

$$\left[\left(\frac{a}{c}\right)^{1/2} \sin\left(\frac{i\pi}{n+1}\right), \dots, \left(\frac{a}{c}\right)^{n/2} \sin\left(\frac{ni\pi}{n+1}\right)\right]^T;$$

2. the eigenvalues and eigenvectors of $\mathrm{Circ}_n(a,b,c)$ are, for $\omega = \exp(\frac{2\pi\sqrt{-1}}{n})$ and for $i \in \{1,\dots,n\}$,

$$b + (a+c)\cos\left(\frac{i2\pi}{n}\right) + \sqrt{-1}(c-a)\sin\left(\frac{i2\pi}{n}\right), \text{ and}$$

$$\left[1,\ \omega^i,\ \cdots,\ \omega^{(n-1)i}\right]^T.$$

Tridiagonal Toeplitz and circulant systems

Let $n \geq 2$, $\epsilon \in]0,1[$, and $a, b, c \in \mathbb{R}$. Let $x, y \colon \mathbb{N}_0 \to \mathbb{R}^n$ solve:

$$x(\ell+1) = \mathrm{Trid}_n(a,b,c)\, x(\ell), \qquad x(0) = x_0,$$
$$y(\ell+1) = \mathrm{Circ}_n(a,b,c)\, y(\ell), \qquad y(0) = y_0.$$

1. if $a = c \neq 0$ and $|b| + 2|a| = 1$, then $\lim_{\ell \to +\infty} x(\ell) = \mathbf{0}$, and the maximum time required for $\|x(\ell)\|_2 \leq \epsilon \|x_0\|_2$ is $\Theta(n^2 \log \epsilon^{-1})$;

2. if $a \neq 0$, $c = 0$ and $0 < |b| < 1$, then $\lim_{\ell \to +\infty} x(\ell) = \mathbf{0}$, and the maximum time required for $\|x(\ell)\|_2 \leq \epsilon \|x_0\|_2$ is $O(n \log n + \log \epsilon^{-1})$;

3. if $a \geq 0$, $c \geq 0$, $b > 0$, and $a + b + c = 1$, then $\lim_{\ell \to +\infty} y(\ell) = y_{\mathrm{ave}}\mathbf{1}$, where $y_{\mathrm{ave}} = \frac{1}{n}\mathbf{1}^T y_0$, and the maximum time required for $\|y(\ell) - y_{\mathrm{ave}}\mathbf{1}\|_2 \leq \epsilon \|y_0 - y_{\mathrm{ave}}\mathbf{1}\|_2$ is $\Theta(n^2 \log \epsilon^{-1})$.

Outline

Deployment, coverage and partitioning

References

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4. F. R. Adler and D. M. Gordon. Optimization, conflict, and nonoverlapping foraging ranges in ants. *American Naturalist*, 162(5):529–543, 2003

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7. S. Martínez, J. Cortés, and F. Bullo. Motion coordination with distributed information. *IEEE Control Systems Magazine*, 27(4):75–88, 2007
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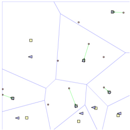
Deployment, coverage and partitioning

Optimize: space partitioning, task allocation, sensor placement

Dynamic vehicle routing

- customers appear randomly space/time
- robots know locations and provide service
- goal: minimize wait time

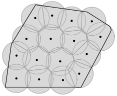
(Pavone, Frazzoli & Bullo; CDC'07 and TAC'09)



Random field estimation

- sensornet estimates spatial stochastic process
- kriging statistical techniques
- goal: minimize error variance

(Graham & Cortés; TAC'09)

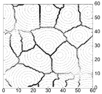


Coverage optimization

DESIGN of performance metrics

1. how to cover a region with n minimum-radius overlapping disks?
2. how to design a minimum-distortion (fixed-rate) vector quantizer?
3. where to place mailboxes in a city / cache servers on the internet?

ANALYSIS of cooperative distributed behaviors



4. how do animals share territory? how do they decide foraging ranges? how do they decide nest locations?
5. what if each vehicle goes to center of mass of own dominance region?
6. what if each vehicle moves away from closest vehicle?

Multi-center functions

- place n robots at $p = \{p_1, \dots, p_n\}$
- partition environment into $W = \{W_1, \dots, W_n\}$
- define expected wait time:

$$\mathcal{H}_{\text{exp}}(p, W) = \int_{W_1} \|q - p_1\| dq + \dots + \int_{W_n} \|q - p_n\| dq$$

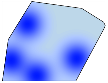
- or more generally

$$\mathcal{H}_{\text{exp}}(p, W) = \sum_{i=1}^{n} \int_{W_i} f(\|q - p_i\|_2) \phi(q) dq$$

where:

$\phi \colon \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ density

$f \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ non-decreasing and piecewise continuously differentiable, possibly with finite jump discontinuities

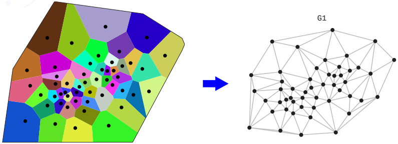


Voronoi partitions

Let $(p_1, \dots, p_n) \in Q^n$ denote the positions of n points

The **Voronoi partition** $\mathcal{V}(P) = \{V_1, \dots, V_n\}$ generated by $(p_1, \dots, p_n)$

$$\begin{aligned} V_i &= \{q \in Q \mid \ \|q - p_i\| \le \|q - p_j\|, \ \forall j \neq i\} \\ &= Q \cap_j \mathcal{HP}(p_i, p_j) \qquad \text{where } \mathcal{HP}(p_i, p_j) \text{ is half plane } (p_i, p_j) \end{aligned}$$

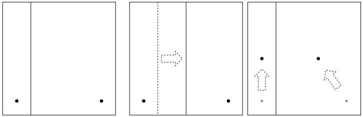


Optimality conditions

$$\mathcal{H}_{\text{exp}}(p, W) = \sum_{i=1}^{n} \int_{W_i} f(\|q - p_i\|_2)\phi(q)dq$$

Theorem (Lloyd '57 "least-square quantization")

1. *at fixed positions, optimal partition is Voronoi*
2. *at fixed partition, optimal positions are "centroids"*
3. *alternate W-p optimization leads to local optimum*



Variety of scenarios

In terms of Voronoi partition,

$$\mathcal{H}_{\text{exp}}(p) = \sum_{i=1}^{n} \int_{V_i(P)} f(\|q - p_i\|_2)\phi(q)dq$$

Distortion problem

$f(x) = x^2$ gives rise to $(\mathsf{J}(W,p)$ moment of inertia and $\mathsf{CM}(W)$ center of mass)

$$\mathcal{H}_{\text{dist}} = \sum_{i=1}^{n} \mathsf{J}(V_i, p_i) \quad = \sum_{i=1}^{n} \mathsf{J}(V_i, \mathsf{CM}(V_i)) + \sum_{i=1}^{n} \mathsf{area}_\phi(V_i)\|p_i - \mathsf{CM}(V_i)\|_2^2$$

Area problem

$f(x) = -1_{[0,a]}(x)$, $a \in \mathbb{R}_{>0}$ gives rise to

$$\mathcal{H}_{\text{area},a}(p) = -\sum_{i=1}^{n} \mathsf{area}_\phi(V_i(P) \cap \overline{B}(p_i, a))$$

Gradient of $\mathcal{H}_{\text{exp}}$ is distributed

For f smooth

$$\begin{aligned} \frac{\partial \mathcal{H}_{\text{exp}}}{\partial p_i}(P) &= \int_{V_i(P)} \frac{\partial}{\partial p_i} f(\|q - p_i\|)\phi(q)dq \\ &+ \int_{\partial V_i(P)} f(\|q - p_i\|)\langle n_i(q), \frac{\partial q}{\partial p_i}\rangle \phi(q)dq \\ &+ \underbrace{\sum_{j \text{ neigh } i} \int_{V_j(P) \cap V_i(P)} f(\|q - p_j\|)\langle n_{ji}(q), \frac{\partial q}{\partial p_i}\rangle \phi(q)dq}_{\text{contrib from neighbors}} \end{aligned}$$

Gradient of $\mathcal{H}_{\text{exp}}$ is distributed

For f smooth

$$\frac{\partial \mathcal{H}_{\text{exp}}}{\partial p_i}(P) = \int_{V_i(P)} \frac{\partial}{\partial p_i} f\left(\|q - p_i\|\right) \phi(q) dq$$
$$+ \int_{\partial V_i(P)} f\left(\|q - p_i\|\right) \langle n_i(q), \frac{\partial q}{\partial p_i} \rangle \phi(q) dq$$
$$- \int_{\partial V_i(P)} f\left(\|q - p_i\|\right) \langle n_i(q), \frac{\partial q}{\partial p_i} \rangle \phi(q) dq$$

Therefore,

$$\frac{\partial \mathcal{H}_{\text{exp}}}{\partial p_i}(P) = \int_{V_i(P)} \frac{\partial}{\partial p_i} f\left(\|q - p_i\|\right) \phi(q) dq$$

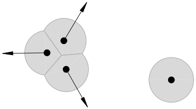
Particular gradients

Distortion problem: continuous performance,

$$\frac{\partial \mathcal{H}_{\text{dist}}}{\partial p_i}(P) = 2\,\mathsf{area}_\phi(V_i(P))(\mathsf{CM}(V_i(P)) - p_i)$$

Area problem: performance has single discontinuity,

$$\frac{\partial \mathcal{H}_{\text{area},a}}{\partial p_i}(P) = \int_{V_i(P) \cap \partial \overline{B}(p_i,a)} \mathsf{n}_{\text{out},\overline{B}(p_i,a)}(q) \phi(q) dq$$



Smoothness properties of $\mathcal{H}_{\text{exp}}$

$\mathsf{Dscn}(f)$ (finite) discontinuities of f

f_- and f_+, limiting values from the left and from the right

Theorem

Expected-value multicenter function $\mathcal{H}_{\text{exp}} : S^n \to \mathbb{R}$ is

1. *globally Lipschitz on S^n; and*
2. *continuously differentiable on $S^n \setminus \mathcal{S}_{\text{coinc}}$, where*

$$\frac{\partial \mathcal{H}_{\text{exp}}}{\partial p_i}(P) = \int_{V_i(P)} \frac{\partial}{\partial p_i} f(\|q - p_i\|_2) \phi(q) dq$$
$$+ \sum_{a \in \mathsf{Dscn}(f)} (f_-(a) - f_+(a)) \int_{V_i(P) \cap \partial \overline{B}(p_i,a)} \mathsf{n}_{\text{out},\overline{B}(p_i,a)}(q) \phi(q) dq$$
$$= \textit{integral over } V_i \; + \; \textit{integral along arcs in } V_i$$

Therefore, the gradient of $\mathcal{H}_{\text{exp}}$ is spatially distributed over $\mathcal{G}_{\text{D}}$

Outline

Geometric-center laws

Uniform networks $\mathcal{S}_{\mathrm{D}}$ and $\mathcal{S}_{\mathrm{LD}}$ of locally-connected first-order agents in a polytope $Q \subset \mathbb{R}^d$ with the Delaunay and r-limited Delaunay graphs as communication graphs

All laws share similar structure

At each communication round each agent performs:

- *it transmits its position and receives its neighbors' positions;*
- *it computes a notion of geometric center of its own cell determined according to some notion of partition of the environment*

Between communication rounds, each robot moves toward this center

VRN-CNTRD ALGORITHM

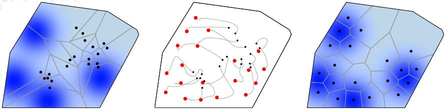
Optimizes distortion $\mathcal{H}_{\mathrm{dist}}$

`Robotic Network:` $\mathcal{S}_{\mathrm{D}}$ in Q, with absolute sensing of own position
`Distributed Algorithm:` VRN-CNTRD
`Alphabet:` $L = \mathbb{R}^d \cup \{\texttt{null}\}$

`function` msg(p, i)
return p

`function` ctrl(p, y)
$V := Q \cap \left(\bigcap \{ H_{p, p_{\mathrm{rcvd}}} \mid \text{for all non-null } p_{\mathrm{rcvd}} \in y \} \right)$
return $\mathsf{CM}(V) - p$

Simulation



initial configuration | gradient descent | final configuration

For $\epsilon \in \mathbb{R}_{>0}$, the *$\epsilon$-distortion deployment task*

$$\mathcal{T}_{\epsilon\text{-distor-dply}}(P) = \begin{cases} \texttt{true}, & \text{if } \left\| p^{[i]} - \mathsf{CM}(V^{[i]}(P)) \right\|_2 \leq \epsilon, \ i \in \{1, \dots, n\}, \\ \texttt{false}, & \text{otherwise} \end{cases}$$

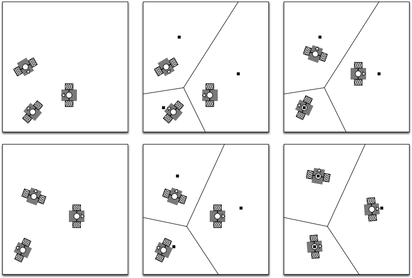
Voronoi-centroid law on planar vehicles

`Robotic Network:` $\mathcal{S}_{\mathrm{vehicles}}$ in Q with absolute sensing of own position
`Distributed Algorithm:` VRN-CNTRD-DYNMCS
`Alphabet:` $L = \mathbb{R}^2 \cup \{\texttt{null}\}$

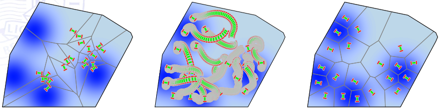
`function` msg$((p, \theta), i)$
return p

`function` ctrl$((p, \theta), (p_{\mathrm{smpld}}, \theta_{\mathrm{smpld}}), y)$
$V := Q \cap \left(\bigcap \{ H_{p_{\mathrm{smpld}}, p_{\mathrm{rcvd}}} \mid \text{for all non-null } p_{\mathrm{rcvd}} \in y \} \right)$
$v := -k_{\mathrm{prop}} (\cos\theta, \sin\theta) \cdot (p - \mathsf{CM}(V))$
$\omega := 2 k_{\mathrm{prop}} \arctan \dfrac{(-\sin\theta, \cos\theta) \cdot (p - \mathsf{CM}(V))}{(\cos\theta, \sin\theta) \cdot (p - \mathsf{CM}(V))}$
return (v, ω)

Algorithm illustration



Simulation



initial configuration — gradient descent — final configuration

LMTD-VRN-NRML algorithm

Optimizes area $\mathcal{H}_{\text{area},\frac{r}{2}}$

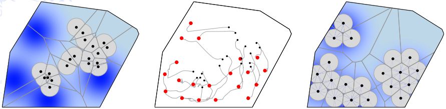
`Robotic Network:` $\mathcal{S}_{\text{LD}}$ in Q with absolute sensing of own position and with communication range r

`Distributed Algorithm:` LMTD-VRN-NRML
`Alphabet:` $L = \mathbb{R}^d \cup \{$`null`$\}$

`function` $\text{msg}(p, i)$
return p

`function` $\text{ctrl}(p, y)$
$V := Q \cap \left(\bigcap \{H_{p,p_{\text{rcvd}}} \mid \text{for all non-null } p_{\text{rcvd}} \in y\}\right)$
$v := \int_{V \cap \partial \overline{B}(p,\frac{r}{2})} \mathsf{n}_{\text{out},\overline{B}(p,\frac{r}{2})}(q)\phi(q)dq$
$\lambda_* := \max\left\{\lambda \mid \delta \mapsto \int_{V \cap \overline{B}(p+\delta v,\frac{r}{2})} \phi(q)dq \text{ is strictly increasing on } [0,\lambda]\right\}$
return $\lambda_* v$

Simulation



initial configuration — gradient descent — final configuration

For $r, \epsilon \in \mathbb{R}_{>0}$,

$$\mathcal{T}_{\epsilon\text{-}r\text{-area-dply}}(P) = \begin{cases} \texttt{true}, & \text{if } \left\| \int_{V^{[i]}(P) \cap \partial \overline{B}(p^{[i]},\frac{r}{2})} \mathsf{n}_{\text{out},\overline{B}(p^{[i]},\frac{r}{2})}(q)\phi(q)dq \right\|_2 \le \epsilon, \ i \in \{1,\dots,n\}, \\ \texttt{false}, & \text{otherwise.} \end{cases}$$

Correctness of the geometric-center algorithms

Theorem

For $d \in \mathbb{N}$, $r \in \mathbb{R}_{>0}$ and $\epsilon \in \mathbb{R}_{>0}$, the following statements hold.

1. *on the network $\mathcal{S}_{\mathrm{D}}$, the law $\mathcal{CC}_{\text{VRN-CNTRD}}$ achieves the ϵ-distortion deployment task $\mathcal{T}_{\epsilon\text{-distor-dply}}$. Moreover, any execution monotonically optimizes $\mathcal{H}_{\mathrm{dist}}$*
2. *on the network $\mathcal{S}_{\text{vehicles}}$, the law $\mathcal{CC}_{\text{VRN-CNTRD-DYNMCS}}$ achieves the ϵ-distortion deployment task $\mathcal{T}_{\epsilon\text{-distor-dply}}$. Moreover, any execution monotonically optimizes $\mathcal{H}_{\mathrm{dist}}$*
3. *on the network $\mathcal{S}_{\mathrm{LD}}$, the law $\mathcal{CC}_{\text{LMTD-VRN-NRML}}$ achieves the ϵ-r-area deployment task $\mathcal{T}_{\epsilon\text{-}r\text{-area-dply}}$. Moreover, any execution monotonically optimizes $\mathcal{H}_{\mathrm{area},\frac{r}{2}}$*

Time complexity of $\mathcal{CC}_{\text{LMTD-VRN-CNTRD}}$

Assume $\mathrm{diam}(Q)$ is independent of n, r and ϵ

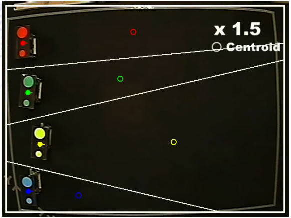
Theorem (Time complexity of LMTD-VRN-CNTRD law)

Assume the robots evolve in a closed interval $Q \subset \mathbb{R}$, that is, $d = 1$, and assume that the density is uniform, that is, $\phi \equiv 1$. For $r \in \mathbb{R}_{>0}$ and $\epsilon \in \mathbb{R}_{>0}$, on the network $\mathcal{S}_{\mathrm{LD}}$

$$\mathrm{TC}(\mathcal{T}_{\epsilon\text{-}r\text{-distor-area-dply}}, \mathcal{CC}_{\text{LMTD-VRN-CNTRD}}) \in O(n^3 \log(n\epsilon^{-1}))$$

Open problem: characterize complexity of deployment algorithms in higher dimensions

Experimental Territory Partitioning



Takahide Goto, Takeshi Hatanaka, Masayuki Fujita

Tokyo Institute of Technology

Experimental Territory Partitioning

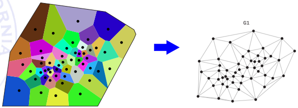
Mac Schwager, Brian Julian, Daniela Rus

Distributed Robots Laboratory, MIT

Outline

1. Models for multi-agent networks
2. Rendezvous and connectivity maintenance
   - Maintaining connectivity
   - Circumcenter algorithms
   - Correctness analysis
   - Time complexity analysis
3. Deployment
   - Multi-center functions
   - Geometric-center laws
   - Peer-to-peer laws
   - Laws for disk-covering and sphere-packing
4. Summary and conclusions

Deployment with minimal communication requirements



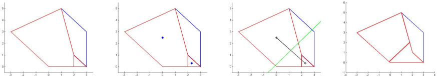
“Voronoi partitioning + move to center” laws require:

1. synchronous & reliable communication
2. communication along edges of “adjacent regions graph”

- is synchrony necessary?
- is it sufficient to communicate peer-to-peer (gossip)?

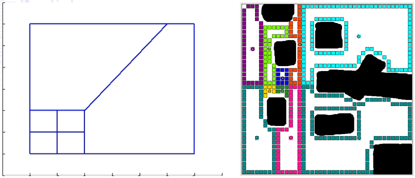
Peer-to-peer partitioning policy

1. Random communication between two regions
2. Compute two centers
3. Compute bisector of centers
4. Partition two regions by bisector

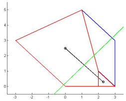


P. Frasca, R. Carli, and F. Bullo. Multiagent coverage algorithms with gossip communication: control systems on the space of partitions. In *American Control Conference*, pages 2228–2235, St. Louis, MO, June 2009

Simulations



Technical Challenges



1. Lyapunov function missing
2. state space is not finite-dimensional
   - non-convex disconnected polygons
   - arbitrary number of vertices
3. peer-to-peer map is not deterministic, ill-defined and discontinuous
   - two regions could have same centroid
   - disconnected/connected discontinuity
4. depending upon communication model, **motion protocol for deterministic/random meetings**

(TC#1) Lyapunov functions for partitions

Standard coverage control

robot i moves towards centroid of its Voronoi region

$$\mathcal{H}_{\exp}(p_1, \dots, p_N) = \sum_{i=1}^{N} \int_{V_i(p_1,\dots,p_N)} f(\|p_i - q\|)\phi(q)dq$$

Peer-to-peer coverage control

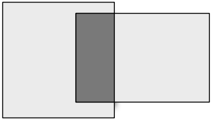
region W_i is modified to appear like a Voronoi region

$$\mathcal{H}_{\exp}(W_1, \dots, W_N) = \sum_{i=1}^{N} \int_{W_i} f(\| \mathsf{CM}(W_i) - q\|)\phi(q)dq$$

(TC#2) Symmetric difference

Given sets A, B, symmetric difference and distance are:

$$A \Delta B = (A \cup B) \setminus (A \cap B), \qquad d_\Delta(A, B) = \mathsf{measure}(A \Delta B)$$



(TC#2) The space of partitions

Definition (space of N-partitions)

$\mathcal{W}$ is collections of N subsets of Q, $v = \{W_i\}_{i=1}^N$, such that

1. $\mathsf{int}(W_i) \cap \mathsf{int}(W_j) = \emptyset$ if $i \neq j$, and
2. $\bigcup_{i=1}^N W_i = Q$
3. **each W_i is closed, has non-empty interior and zero-measure boundary**

Theorem (topological properties of the space of partitions)

*$\mathcal{W}$ with $d_\Delta(u, v) = \sum_{i=1}^N d_\Delta(u_i, W_i)$ is metric and **precompact***

(TC#3) Convergence thm with uniformly persistent switches

- X is metric space
- finite collection of maps $T_i \colon X \to X$ for $i \in I$
- consider a sequence $\{x_\ell\}_{\ell \geq 0} \subset X$ with

$$x_{\ell+1} = T_{i(\ell)}(x_\ell)$$

Assume:

1. $W \subset X$ compact and positively invariant for each T_i
2. $U \colon W \to \mathbb{R}$ decreasing along each T_i
3. U and T_i are continuous on W
4. for all $i \in I$, there are infinite times ℓ such that $x_{\ell+1} = T_i(x_\ell)$ and delay between any two consecutive times is bounded

If $x_0 \in W$, then

$$x_\ell \to \text{(intersection of sets of fixed points of all } T_i) \cap U^{-1}(c)$$

(TC#3) Convergence thm with randomly persistent switches

- finite collection of maps $T_i \colon X \to X$ for $i \in I$
- consider sequences $\{x_\ell\}_{\ell \geq 0} \subset X$ with

$$x_{\ell+1} = T_{i(\ell)}(x_\ell)$$

Assume:

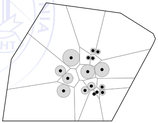
1. $W \subset X$ compact and positively invariant for each T_i
2. $U \colon W \to \mathbb{R}$ decreasing along each T_i
3. U and T_i are continuous on W
4. there exists probability $p \in]0, 1[$ such that, for all indices $i \in I$ and times ℓ, we have $\text{Prob}\,[x_{\ell+1} = T_i(x_\ell) \,|\, \text{past}] \geq p$

If $x_0 \in W$, then almost surely

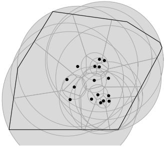
$$x_\ell \to \text{(intersection of sets of fixed points of all } T_i) \cap U^{-1}(c)$$

Outline

Deployment: basic behaviors



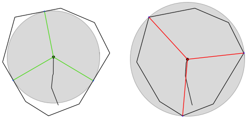
"move away from closest"



"move towards furthest"

Equilibria? Asymptotic behavior?
Optimizing network-wide function?

Deployment: 1-center optimization problems



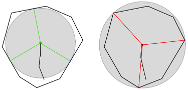
$\mathsf{sm}_Q(p) = \min\{\|p-q\| \mid q \in \partial Q\}$ Lipschitz $\quad 0 \in \partial\,\mathsf{sm}_Q(p) \Leftrightarrow p \in \mathsf{IC}(Q)$

$\mathsf{lg}_Q(p) = \max\{\|p-q\| \mid q \in \partial Q\}$ Lipschitz $\quad 0 \in \partial\,\mathsf{lg}_Q(p) \Leftrightarrow p = \mathsf{CC}(Q)$

Locally Lipschitz function V are differentiable a.e.
Generalized gradient of V is

$$\partial V(x) = \text{convex closure}\{\lim_{i\to\infty} \nabla V(x_i) \mid x_i \to x\,,\ x_i \notin \Omega_V \cup S\}$$

Deployment: 1-center optimization problems



$+$ gradient flow of sm_Q $\quad \dot p_i = +\mathsf{Ln}[\partial\,\mathsf{sm}_Q](p)$ "move away from closest"

$-$ gradient flow of lg_Q $\quad \dot p_i = -\mathsf{Ln}[\partial\,\mathsf{lg}_Q](p)$ "move toward furthest"

For X essentially locally bounded, **Filippov solution** of $\dot x = X(x)$ is absolutely continuous function $t \in [t_0, t_1] \mapsto x(t)$ verifying

$$\dot x \in K[X](x) = \mathsf{co}\{\lim_{i\to\infty} X(x_i) \mid x_i \to x\,,\ x_i \notin S\}$$

For V locally Lipschitz, gradient flow is $\dot x = \mathsf{Ln}[\partial V](x)$
Ln = least norm operator

Nonsmooth LaSalle Invariance Principle

Evolution of V along Filippov solution $t \mapsto V(x(t))$ is differentiable a.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) \in \underbrace{\widetilde{\mathcal{L}}_X V(x(t)) = \{a \in \mathbb{R} \mid \exists v \in K[X](x) \text{ s.t. } \zeta \cdot v = a\,,\ \forall \zeta \in \partial V(x)\}}_{\text{set-valued Lie derivative}}$$

LaSalle Invariance Principle

For S compact and strongly invariant with $\max \widetilde{\mathcal{L}}_X V(x) \le 0$

Any Filippov solution starting in S converges to largest weakly invariant set contained in $\left\{x \in S \mid 0 \in \widetilde{\mathcal{L}}_X V(x)\right\}$

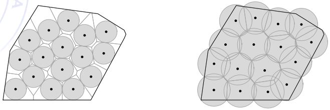
E.g., **nonsmooth gradient flow** $\dot x = -\mathsf{Ln}[\partial V](x)$ converges to critical set

Deployment: multi-center optimization

sphere packing and disk covering

"move away from closest": $\quad \dot p_i = +\mathsf{Ln}(\partial\,\mathsf{sm}_{V_i(P)})(p_i)$ — at fixed $V_i(P)$

"move towards furthest": $\quad \dot p_i = -\mathsf{Ln}(\partial\,\mathsf{lg}_{V_i(P)})(p_i)$ — at fixed $V_i(P)$



Aggregate objective functions!

$$\mathcal{H}_{\text{sp}}(P) = \min_i \mathsf{sm}_{V_i(P)}(p_i) = \min_{i\neq j}\left[\tfrac{1}{2}\|p_i - p_j\|, \mathsf{dist}(p_i, \partial Q)\right]$$

$$\mathcal{H}_{\text{dc}}(P) = \max_i \mathsf{lg}_{V_i(P)}(p_i) = \max_{q\in Q}\left[\min_i \|q - p_i\|\right]$$

Deployment: multi-center optimization

Critical points of $\mathcal{H}_{sp}$ and $\mathcal{H}_{dc}$ (locally Lipschitz)

- If $0 \in \operatorname{int} \partial\mathcal{H}_{sp}(P)$, then P is strict local maximum, all agents have same cost, and P is **incenter Voronoi configuration**
- If $0 \in \operatorname{int} \partial\mathcal{H}_{dc}(P)$, then P is strict local minimum, all agents have same cost, and P is **circumcenter Voronoi configuration**

Aggregate functions **monotonically optimized** along evolution

$$\min \widetilde{\mathcal{L}}_{\mathsf{Ln}(\partial\,\mathsf{sm}_{\mathcal{V}(P)})}\mathcal{H}_{\mathbf{sp}}(P) \geq 0 \qquad \max \widetilde{\mathcal{L}}_{-\mathsf{Ln}(\partial\,\mathsf{lg}_{\mathcal{V}(P)})}\mathcal{H}_{\mathbf{dc}}(P) \leq 0$$

Asymptotic convergence via nonsmooth LaSalle principle

- Convergence to configurations where all agents whose local cost coincides with aggregate cost are centered
- Convergence to center Voronoi configurations still open

Voronoi-circumcenter algorithm

`Robotic Network:` $\mathcal{S}_{\mathrm{D}}$ in Q with absolute sensing of own position

`Distributed Algorithm:` VRN-CRCMCNTR
`Alphabet:` $L = \mathbb{R}^d \cup \{$`null`$\}$

`function` $\mathrm{msg}(p, i)$
return p

`function` $\mathrm{ctrl}(p, y)$
$V := Q \cap \big(\bigcap\{H_{p,p_{\mathrm{rcvd}}} \mid \text{for all non-null } p_{\mathrm{rcvd}} \in y\}\big)$
return $\mathsf{CC}(V) - p$

Voronoi-incenter algorithm

`Robotic Network:` $\mathcal{S}_{\mathrm{D}}$ in Q with absolute sensing of own position

`Distributed Algorithm:` VRN-NCNTR
`Alphabet:` $L = \mathbb{R}^d \cup \{$`null`$\}$

`function` $\mathrm{msg}(p, i)$
return p

`function` $\mathrm{ctrl}(p, y)$
$V := Q \cap \big(\bigcap\{H_{p,p_{\mathrm{rcvd}}} \mid \text{for all non-null } p_{\mathrm{rcvd}} \in y\}\big)$
return $x \in \mathsf{IC}(V) - p$

Correctness of the geometric-center algorithms

For $\epsilon \in \mathbb{R}_{>0}$, the *$\epsilon$-disk-covering deployment task*

$$\mathcal{T}_{\epsilon\text{-dc-dply}}(P) = \begin{cases} \texttt{true}, & \text{if } \|p^{[i]} - \mathsf{CC}(V^{[i]}(P))\|_2 \leq \epsilon,\ i \in \{1, \dots, n\}, \\ \texttt{false}, & \text{otherwise,} \end{cases}$$

For $\epsilon \in \mathbb{R}_{>0}$, the *$\epsilon$-sphere-packing deployment task*

$$\mathcal{T}_{\epsilon\text{-sp-dply}}(P) = \begin{cases} \texttt{true}, & \text{if } \mathsf{dist}_2(p^{[i]}, \mathsf{IC}(V^{[i]}(P))) \leq \epsilon,\ i \in \{1, \dots, n\}, \\ \texttt{false}, & \text{otherwise,} \end{cases}$$

Theorem

For $d \in \mathbb{N}$, $r \in \mathbb{R}_{>0}$ and $\epsilon \in \mathbb{R}_{>0}$, the following statements hold.

1. *on the network $\mathcal{S}_{\mathrm{D}}$, any execution of the law $\mathcal{CC}_{\text{VRN-CRCMCNTR}}$ monotonically optimizes the multicenter function $\mathcal{H}_{\mathrm{dc}}$;*
2. *on the network $\mathcal{S}_{\mathrm{D}}$, any execution of the law $\mathcal{CC}_{\text{VRN-NCNTR}}$ monotonically optimizes the multicenter function $\mathcal{H}_{\mathrm{sp}}$.*

Outline

Summary and conclusions

Examined various motion coordination tasks

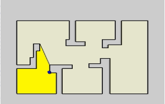
1. **rendezvous:** circumcenter algorithms
2. **connectivity maintenance:** flexible constraint sets in convex/nonconvex scenarios
3. **deployment:** gradient algorithms based on geometric centers

Correctness and **(1-d) complexity analysis** of geometric-center control and communication laws via

1. Discrete- and continuous-time nondeterministic dynamical systems
2. Invariance principles, stability analysis
3. Geometric structures and geometric optimization

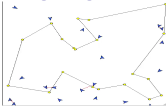
A sample of other coordination problems

Visibility-based deployment



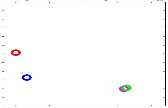
A. Ganguli, J. Cortés, and F. Bullo. Visibility-based multi-agent deployment in orthogonal environments. In *American Control Conference*, pages 3426–3431, New York, July 2007

Target assignment



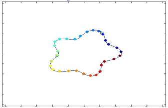
S. L. Smith and F. Bullo. Monotonic target assignment for robotic networks. *IEEE Transactions on Automatic Control*, 54(10), 2009. (Submitted June 2007) to appear

Synchronized patrolling



S. Susca, F. Bullo, and S. Martínez. Synchronization of beads on a ring. In *IEEE Conf. on Decision and Control*, pages 4845–4850, New Orleans, LA, December 2007

Boundary estimation



S. Susca, S. Martínez, and F. Bullo. Monitoring environmental boundaries with a robotic sensor network. *IEEE Transactions on Control Systems Technology*, 16(2):288–296, 2008

Emerging Motion Coordination Discipline

1. **network modeling**
   network, ctrl+comm algorithm, task, complexiy
   coordination algorithm
   optimal deployment, rendezvous
   adaptive, scalable, asynchronous, agent arrival/departure
2. **Systematic algorithm design**
   - meaningful aggregate cost functions
   - geometric structures
   - stability theory for networked hybrid systems
3. Literature full of exciting problems, solutions, and tools:
   Formation control, consensus, cohesiveness, flocking, collective synchronization, boundary estimation, cooperative control over constant graphs, quantization, asynchronism, delays, distributed estimation, spatial estimation, data fusion, target tracking, networks with minimal capabilities, target assignment, vehicle dynamics and energy-constrained motion, vehicle routing, dynamic servicing problems, load balancing, robotic implementations,...