

Lecture notes for Math 439

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These are lecture notes designed using the book

Geometric Control of Mechanical Systems

Francesco Bullo and Andrew D. Lewis

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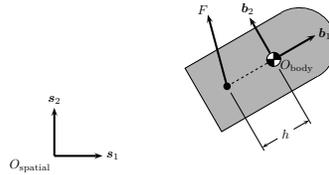
The material covered in these notes is selected from

1. Sections 3.2 and 4.1,
2. Section 3.3,
3. Section 3.5,
4. Sections 2.3 and 3.6,
5. Section 4.2,
6. Sections 3.8 and 4.3,
7. Section 4.4,
8. Section 4.6,
9. Section 7.1 (not much from here),
10. Section 8.3,
11. Section 13.1,
12. Section 13.2.

These lecture notes are more conversational than the text, and therefore somewhat less precise. Pedants please read the book.

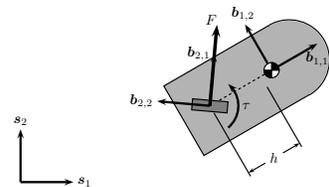
1. Introduction—what will you learn in this class?

- Consider the following example of a hovercraft in the plane:



- The force F can have its direction and magnitude arbitrarily specified, and the body moves on the plane without friction.
- Problems:
 - Can one steer between arbitrary configurations at rest?
 - How* can one steer between arbitrary configurations at rest?
 - If the motor rotating the fan angle breaks, what is the answer to the preceding two questions?

- In practice, the dynamics of the fan are significant, and so have to be modelled:



- With the additional dynamics, what are the answers to the first two questions?
- Answers:
 - Yes. This is not a completely trivial problem, but we will learn to answer it in this class.
 - The answer to 1 that we will give is, in fact, constructive, so we will also answer 2.

3. The answer to 3 depends on the angle the fan is pointing when the motor breaks.
 - (a) If the fan is pointing straight ahead or straight back, then it is clear that the answer to 1 is, “No.” Indeed, one can only reach configurations that are aligned with the initial configuration.
 - (b) At any other fan angle, the answer is not so clear, and in fact is not perfectly understood in the literature. What *is* true is that one cannot steer to all configurations near the initial configuration without making large excursions.
4. Here the question becomes surprisingly complicated. The answer to 1 is, “No,” provided the question is posed clearly.

- What is the point?
- This example is a “simple” one in the sense that the equations of motion are not exceedingly complicated. And the questions we are asking are basic ones. Nonetheless, they are nontrivial to answer.
- One way to answer these questions is to understand very well the models involved. That is what this class is about.
- We will understand quite precisely the ideas of “configuration,” “velocity,” “force,” “acceleration,” etc.

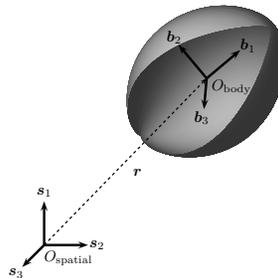
2. Configuration spaces and differentiable manifolds

- We now begin skipping between Chapters 3 and 4. Initially, we consider Sections 3.2 and 4.1 together.

2.1. Configuration space

- A *particle* is an object with mass concentrated at a point.
- A *rigid body* is an object with mass and volume.
- We will be more careful with these “definitions” later.
- A *free mechanical system* is a collection P_1, \dots, P_{N_P} of particles and B_1, \dots, B_{N_B} of rigid bodies which move independently of one another.
- How do we specify a configuration of a free mechanical system?
- We do so by specifying separately the configuration of each particle, and each rigid body.

- To specify the location of a particle, choose an inertial reference frame $(O_{\text{spatial}}, \{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\})$ consisting of a spatial origin O_{spatial} and an orthonormal frame $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ at O_{spatial} . The position of the particle P_j is exactly determined by a vector $\mathbf{r}_j \in \mathbb{R}^3$ from O_{spatial} to the location of P_j .
- To specify the position of a body, additionally specify a body reference frame $(O_{\text{body}}, \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\})$ that is fixed to move with the body.



- The body B_j is specified by the vector $\mathbf{r}_j = O_{\text{body}} - O_{\text{spatial}} \in \mathbb{R}^3$, along with a specification of the orientation of the orthonormal frame $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ relative to $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$.
- This orientation is determined by specifying the components of \mathbf{b}_a , $a \in \{1, 2, 3\}$, relative to the basis $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$:

$$\mathbf{b}_a = R_{1a}\mathbf{s}_1 + R_{2a}\mathbf{s}_2 + R_{3a}\mathbf{s}_3.$$

- The matrix

$$\mathbf{R}_j = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$

is an orthogonal matrix since the bases $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ are orthonormal.

- If one chooses the bases to have the same orientation (say, right-handed), then $\mathbf{R}_j \in \text{SO}(3)$, where

$$\text{SO}(3) = \left\{ \mathbf{R} \in \mathbb{R}^{3 \times 3} \mid \mathbf{R}\mathbf{R}^T = \mathbf{I}_3, \det \mathbf{R} = 1 \right\}$$

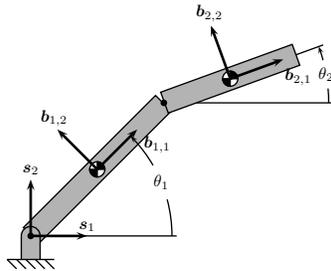
is the *special orthogonal group* in three-dimensions. (Removing the condition that $\det \mathbf{R} = 1$ gives $\text{O}(3)$, the *orthogonal group*.)

- *Punchline:* To specify the configuration of a rigid body one specifies a point in $\text{SO}(3) \times \mathbb{R}^3$. Therefore, the configuration of a free mechanical system is specified by a point in

$$\mathbf{Q}_{\text{free}} = \underbrace{\mathbb{R}^3 \times \cdots \times \mathbb{R}^3}_{N_P \text{ copies}} \times \underbrace{(\text{SO}(3) \times \mathbb{R}^3) \times \cdots \times (\text{SO}(3) \times \mathbb{R}^3)}_{N_B \text{ copies}}.$$

- Now suppose we have interconnections between particles and bodies, as is almost always the case.

Example 2.1 Two-link planar manipulator:



Here we have

$$Q_{\text{free}} = (\text{SO}(3) \times \mathbb{R}^3) \times (\text{SO}(3) \times \mathbb{R}^3).$$

However, the actual configurations of the system are specified by the angles θ_1 and θ_2 as shown in the figure. Where do these angles live? Each angle is essentially a number, keeping in mind that, if two numbers differ by an integer multiple of 2π , then they are really the same angle. More concretely, each angle is measured by a

point on the circle

$$\mathbb{S}^1 = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}.$$

Thus the configurations of this simple two-link robot are specified by a point in $\mathbb{S}^1 \times \mathbb{S}^1$. □

- Issues arising from these considerations:
 1. How does one do calculations on configuration spaces? We have only been taught to do calculations on \mathbb{R}^n , or open subsets of \mathbb{R}^n .
 2. In practice, one simply “chooses coordinates” for the configuration space, and then pretends that one is then working with Euclidean space. This is fine, but one should be sure one talks about things that do not depend on a particular choice of coordinates.
 3. To get around these issues, we dig into differential geometry. This will take several weeks.

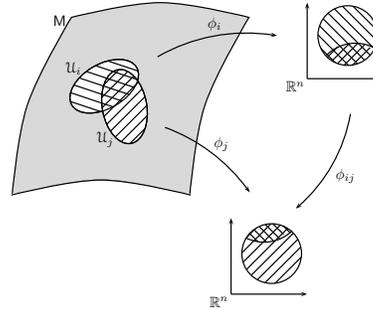
Configuration spaces and differentiable manifolds (cont'd)

Recall: a *diffeomorphism* is a bijection $f: \mathcal{U} \rightarrow \mathcal{V}$ between open subsets $\mathcal{U}, \mathcal{V} \subset \mathbb{R}^n$ which is infinitely differentiable, and for which the inverse is infinitely differentiable.

2.2. Charts and atlases

Definition 2.2 Let S be a set. A *chart* for S is a pair (\mathcal{U}, ϕ) with

- (i) \mathcal{U} a subset of S and
- (ii) $\phi: \mathcal{U} \rightarrow \mathbb{R}^n$ an injection for which $\phi(\mathcal{U})$ is an open subset of \mathbb{R}^n .



An *atlas* for S is a collection $\mathcal{A} = \{(\mathcal{U}_a, \phi_a)\}_{a \in A}$ of charts with the property $S = \cup_{a \in A} \mathcal{U}_a$, and such that whenever $\mathcal{U}_a \cap \mathcal{U}_b \neq \emptyset$ we have

- (iii) $\phi_a(\mathcal{U}_a \cap \mathcal{U}_b)$ and $\phi_b(\mathcal{U}_a \cap \mathcal{U}_b)$ are open subsets of \mathbb{R}^n ,
- (iv) $\phi_{ab} \triangleq \phi_b \circ \phi_a^{-1}|_{\phi_a(\mathcal{U}_a \cap \mathcal{U}_b)}$ is a diffeomorphism from $\phi_a(\mathcal{U}_a \cap \mathcal{U}_b)$ to $\phi_b(\mathcal{U}_a \cap \mathcal{U}_b)$. □

- *Idea:* A chart parameterizes a subset of the set S . The *overlap condition* (iv) ensures that different parameterizations will be compatible.

Example 2.3 Let $S = \mathbb{R}^2$ and define charts (\mathcal{U}_1, ϕ_1) and (\mathcal{U}_2, ϕ_2) by $\mathcal{U}_1 = S$ and $\phi_1(x, y) = (x, y)$, and

$$\mathcal{U}_2 = \mathbb{R}^2 \setminus \{(x, 0) \mid x \leq 0\}, \quad \phi_2(x, y) = (\sqrt{x^2 + y^2}, \text{atan}(x, y)).$$

The latter are *polar coordinates*, with which you are familiar, but now we are being more formal.

We now verify the overlap condition. Note that

$$\begin{aligned} \phi_1(\mathcal{U}_1 \cap \mathcal{U}_2) &= \mathbb{R}^2 \setminus \{(x, 0) \mid x \leq 0\}, \\ \phi_2(\mathcal{U}_1 \cap \mathcal{U}_2) &= \{(r, \theta) \mid r > 0, \theta \in]-\pi, \pi[\}. \end{aligned}$$

One computes

$$\phi_{12}(x, y) = (\sqrt{x^2 + y^2}, \text{atan}(x, y))$$

which has inverse

$$\phi_{12}^{-1}(r, \theta) = (r \cos \theta, r \sin \theta).$$

The Jacobians of these maps are

$$D\phi_{12}(x, y) = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix}, \quad D\phi_{12}^{-1}(r, \theta) = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}.$$

Since the entries in these matrices are continuous, the overlap map and its inverse are differentiable. In fact, both maps are infinitely differentiable, so the overlap condition is satisfied. \square

Example 2.4 Take $S = \mathbb{S}^1 \subset \mathbb{R}^2$. On \mathbb{S}^1 we define two charts (\mathcal{U}_1, ϕ_1) and (\mathcal{U}_2, ϕ_2) by

$$\begin{aligned} \mathcal{U}_1 &= \mathbb{S}^1 \setminus \{(-1, 0)\}, & \phi_1(x, y) &= \text{atan}(x, y) \\ \mathcal{U}_2 &= \mathbb{S}^1 \setminus \{(1, 0)\}, & \phi_2(x, y) &= \text{atan}(-x, -y). \end{aligned}$$

Thus ϕ_1 measures the angle (denote it by θ_1) of the point (x, y) from the positive x -axis, and ϕ_2 measures the angle (denote it by θ_2) of the point (x, y) from the negative x -axis. Note that $\phi_1(\mathcal{U}_1) =]-\pi, \pi[$ and $\phi_2(\mathcal{U}_2) =]-\pi, \pi[$. Note also that $\phi_1(\mathcal{U}_1 \cap \mathcal{U}_2) =]-\pi, 0[\cup]0, \pi[$, and that $\phi_2(\mathcal{U}_1 \cap \mathcal{U}_2) =]-\pi, 0[\cup]0, \pi[$. One

computes the overlap maps as

$$\phi_{12}(\theta_1) = \begin{cases} \theta_1 - \pi, & \theta_1 \in]0, \pi[\\ \pi + \theta_1, & \theta_1 \in]-\pi, 0[\end{cases}, \quad \phi_{12}(\theta_2) = \begin{cases} \theta_2 - \pi, & \theta_2 \in]0, \pi[\\ \pi + \theta_2, & \theta_2 \in]-\pi, 0[\end{cases}.$$

These maps are clearly inverses of one another, and are also clearly infinitely differentiable. Thus this gives an atlas for \mathbb{S}^1 .

Remark 2.5 It is not possible to cover \mathbb{S}^1 with a single chart. It is not obvious why this is so, but it can be roughly argued as follows. A chart for \mathbb{S}^1 , by definition, takes values in an open subset of \mathbb{R} . An open subset of \mathbb{R} is a collection of open intervals. Now, there exists a curve in \mathbb{S}^1 that ends up where it started, without intersecting itself (i.e., the curve $[0, 2\pi] \ni t \mapsto (\cos t, \sin t) \in \mathbb{S}^1$). No such curve is possible in a collection of open intervals. Thus, one needs at least two charts to cover \mathbb{S}^1 . \square

Configuration spaces and differentiable manifolds (cont'd)

2.3. Manifolds

Definition 2.6 Two atlases $\mathcal{A}_1 = \{(U_a, \phi_a)\}_{a \in A}$ and $\mathcal{A}_2 = \{(V_b, \psi_b)\}_{b \in B}$ for a set S are *equivalent* if $\mathcal{A}_1 \cup \mathcal{A}_2$ is an atlas. □

- *Idea*: Charts from different atlases must satisfy the overlap condition relative to one another.

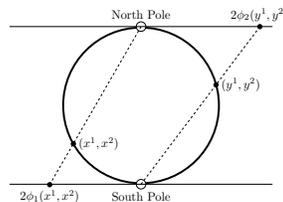
Definition 2.7 A *differentiable structure* on a set S is an equivalence class of atlases with the preceding equivalence relation. A *manifold* is a pair (S, \mathcal{D}) where \mathcal{D} is a differentiable structure on S . □

- To specify a differentiable structure in practice, one simply specifies *some* atlas, and then considers the equivalence class corresponding to this, usually without thinking about it, as in our examples.
- In practice, one writes a typical manifold as M , regarding the differentiable structure as having been fixed.

- If all charts for a manifold take value in \mathbb{R}^n for a fixed n , then $\dim(M) = n$ is the *dimension* of M .

Examples 2.8 1. \mathbb{R}^n is a differentiable manifold with the natural differentiable structure defined by the atlas $\{(\mathbb{R}^n, \text{id}_{\mathbb{R}^n})\}$.

2. More generally, if $\mathcal{U} \subset \mathbb{R}^n$ is an open set, then this has the differentiable structure defined by the atlas $(\mathcal{U}, \text{id}_{\mathbb{R}^n}|_{\mathcal{U}})$.
3. \mathbb{S}^n is a differentiable manifold, with an atlas defined by stereographic projection:



See text for details.

4. On \mathbb{S}^2 one can also specify longitude/latitude coordinates, for example. It is an exercise to show that the coordinate chart is (\mathcal{V}, ψ) where

$$\mathcal{V} = \mathbb{S}^2 \setminus \{(x, 0, z) \mid x \leq 0\}, \quad \psi(x, y, z) = (\text{atan}(x, y), \text{acos}(z)) = (\theta, \phi),$$

defining the coordinates (θ, ϕ) , where θ measures longitude in the range $] - \pi, \pi[$ and ϕ measures latitude in the range $]0, \pi[$. \square

2.4. Back to configuration spaces

- To parameterize configuration space, we need to parameterize $SO(3)$, since this appears in the free configuration space of a rigid body.

- Let us define

$$SO(n) = \left\{ \mathbf{R} \in \mathbb{R}^{n \times n} \mid \mathbf{R}\mathbf{R}^T = \mathbf{I}_n, \det \mathbf{R} = 1 \right\},$$

and think about parameterizing $SO(n)$.

- The case $n = 2$ can be understood. A general 2×2 matrix looks like

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Note that $\mathbb{R}^{2 \times 2}$ is naturally a 4-dimensional manifold. A general *special*

orthogonal 2×2 matrix looks like

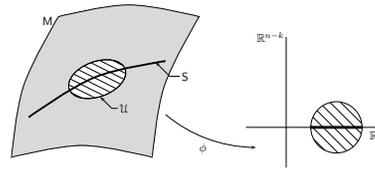
$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

for some $\theta \in \mathbb{R}$. Note that values of θ differing by an integer multiple of 2π produce the same orthogonal matrix. Thus, really, $SO(2)$ looks like a copy of \mathbb{S}^1 sitting in $\mathbb{R}^{2 \times 2}$.

- It is not so perfectly clear how to similarly parameterize $SO(3) \subset \mathbb{R}^{3 \times 3}$. However, since we understand $\mathbb{R}^{3 \times 3}$ well (it is naturally a 9-dimensional manifold), maybe we would be better off understanding $SO(3)$ as it sits inside $\mathbb{R}^{3 \times 3}$.

2.5. Submanifolds

Definition 2.9 A subset S of a manifold M is a *submanifold* if, for each point $x \in S$, there is an admissible chart (\mathcal{U}, ϕ) with $x \in \mathcal{U}$ and such that



- (i) ϕ takes its values in a product $\mathbb{R}^k \times \mathbb{R}^{n-k}$, and
- (ii) $\phi(\mathcal{U} \cap S) = \phi(\mathcal{U}) \cap (\mathbb{R}^k \times \{0\})$.

A chart with these properties is a *submanifold chart* for S . □

- *Idea:* The coordinates in $\mathbb{R}^k \times \{0\}$ give a chart for S .
- Like all “subobjects” you have encountered, the idea is that the subset acquires the structure of the set within which it sits (e.g., a vector space structure for subspaces, a group structure for subgroups).
- Note that if (\mathcal{U}, ϕ) is a submanifold chart for S , then $(\mathcal{U} \cap S, \phi|_{\mathcal{U} \cap S})$ is a chart. The overlap condition for such charts can be verified. Thus submanifolds are manifolds.

Configuration spaces and differentiable manifolds (cont'd)

Example 2.10 Take $M = \mathbb{R}^2$ and $S = \mathbb{S}^1$. Let us find a submanifold chart. Seems reasonable to adapt polar coordinates, since the standard polar coordinate “ r ” has value 1 on S . Thus we shift it:

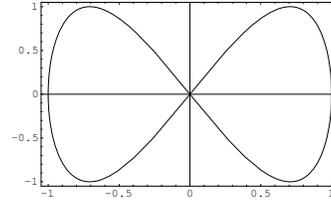
$$\mathcal{U} = \mathbb{R}^2 \setminus \{(x, 0) \mid x \leq 0\}, \quad \phi(x, y) = (\sqrt{x^2 + y^2} - 1, \text{atan}(x, y)).$$

We have $\phi(\mathcal{U}) =]-1, \infty[\times]-\pi, \pi[$, and $\phi(\mathcal{U} \cap S) = \{0\} \times]-\pi, \pi[\subset \mathbb{R}^1 \times \mathbb{R}^1$. Thus this is a submanifold chart. To show that \mathbb{S}^1 is a submanifold of \mathbb{R}^2 , we need an atlas of submanifold charts. In this case, this means we would need to cover the point $(-1, 0) \in \mathbb{S}^1$ with a submanifold chart. This is left as an easy exercise.

Conclusion: \mathbb{S}^1 is a submanifold of \mathbb{R}^2 . □

- Many parameterized curves in \mathbb{R}^2 or \mathbb{R}^3 , or parameterized surfaces in \mathbb{R}^3 , that may have been encountered in vector calculus are indeed submanifolds.
- However, not all subsets are submanifolds, and some “nice” sets may not be.

Example 2.11 Consider the curve in \mathbb{R}^2 defined by the parameterization $t \mapsto (\sin t, \sin(2t))$. Let S be the image of this curve. This subset is not a submanifold. Clearly the sticky point is at the origin. One should imagine trying to show that there is a set of coordinates for \mathbb{R}^2 containing $(0, 0) \in S$ which maps S to one of the coordinate axes. Although it is not so easy to rigorously show why this is impossible, it is easy to imagine why it is. This shows that one needs to exercise some care when talking about all submanifolds, since perhaps not all nice objects of your experience are submanifolds. \square



- **Fact:** $SO(n)$ is a submanifold of $\mathbb{R}^{n \times n}$. This is easy to show using Proposition 3.42 (this is Exercise E3.17), but we will not develop the machinery to prove this.

2.6. Back to configuration spaces

- Previously, we had discussed free mechanical systems. Now we wish to allow interconnections between bodies and particles.

Definition 2.12 An *interconnected mechanical system* is a collection $\{P_\alpha\}_{\alpha \in \{1, \dots, N_P\}} \cup \{B_\beta\}_{\beta \in \{1, \dots, N_B\}}$ of N_P particles and N_B rigid bodies restricted to move on a submanifold Q of Q_{free} . The manifold Q is the *configuration manifold* for the system. \square

- It is possible that one may be interested in mechanical systems for which the set of admissible configurations is *not* a submanifold. For example, a robot which encounters an obstacle in its workspace is of this sort. There will be a discontinuity when the robot hits the obstacle.

Example 2.13 (Planar rigid body) We consider a planar rigid body. Thus $Q_{\text{free}} = SO(3) \times \mathbb{R}^3$. Recall that to assign a configuration of the body to a point in Q_{free} we choose a spatial frame $(O_{\text{spatial}}, \{s_1, s_2, s_3\})$ and a body frame $(O_{\text{body}}, \{b_1, b_2, b_3\})$. We wish to adapt our choice of these frames in a manner suited to the system. Since the body is planar, this means it moves on some two-dimensional plane P . We then choose our frames to meet the following criterion:

1. $O_{\text{spatial}} \in P$;
2. s_3 is orthogonal to P ;
3. O_{body} is at the center of mass of the body, which we assume to be in P ;

4. \mathbf{b}_3 is orthogonal to P .

With these assumptions, a point in Q_{free} that corresponds to a configuration of the body has the form

$$\left(\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, (x, y, 0) \right) \in \text{SO}(3) \times \mathbb{R}^3,$$

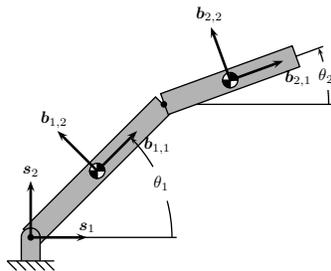
for $x, y, \theta \in \mathbb{R}$. Note that such configurations really live only in $\text{SO}(2) \times \mathbb{R}^2$, and are specified by

$$\left(\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, (x, y) \right) \in \text{SO}(2) \times \mathbb{R}^2.$$

Thus the configuration space is a submanifold of $\text{SO}(3) \times \mathbb{R}^3$ that, in a natural way, looks like $\text{SO}(2) \times \mathbb{R}^2$. However, we have already seen that $\text{SO}(2)$ is essentially \mathbb{S}^1 . Thus we shall take $Q = \mathbb{S}^1 \times \mathbb{R}^2$, with the “angle” in \mathbb{S}^1 measuring the rotation of the body, and the point $(x, y) \in \mathbb{R}^2$ indicating the position of the center of mass. \square

Configuration spaces and differentiable manifolds (cont'd)

Example 2.14 (Two-link manipulator) We return “officially” to our planar two-link robot. We choose a spatial frame $(O_{\text{spatial}}, \{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\})$, and body frames $(O_{1,\text{body}}, \{\mathbf{b}_{1,1}, \mathbf{b}_{1,2}, \mathbf{b}_{1,3}\})$ and $(O_{2,\text{body}}, \{\mathbf{b}_{2,1}, \mathbf{b}_{2,2}, \mathbf{b}_{2,3}\})$ as follows:



Thus we assume

1. $O_{\text{spatial}}, O_{1,\text{body}},$ and $O_{2,\text{body}}$ lie in the plane P of motion of the links,
2. $\mathbf{s}_3, \mathbf{b}_{1,3},$ and $\mathbf{b}_{2,3}$ are orthogonal to P , and
3. $O_{i,\text{body}}$ is at the center of mass of link i , which, for simplicity, we suppose to be at the link's midpoint.

In this case, $Q_{\text{free}} = (SO(3) \times \mathbb{R}^3) \times (SO(3) \times \mathbb{R}^3)$, and we denote a typical point in Q_{free} by $((\mathbf{R}_1, \mathbf{r}_1), (\mathbf{R}_2, \mathbf{r}_2))$. We then easily see that

$$\mathbf{R}_1 = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 \\ \sin \theta_1 & \cos \theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{r}_1 = \left(\frac{\ell_1}{2} \cos \theta_1, \frac{\ell_1}{2} \sin \theta_1, 0\right)$$

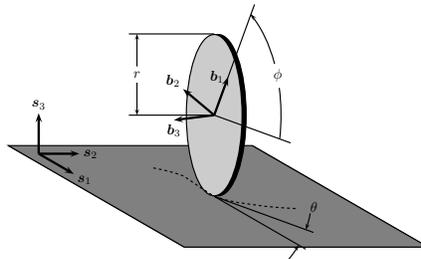
$$\mathbf{R}_2 = \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & 0 \\ \sin \theta_2 & \cos \theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{r}_2 = \left(\ell_1 \cos \theta_1 + \frac{\ell_2}{2} \cos \theta_2, \ell_1 \sin \theta_1 + \frac{\ell_2}{2} \sin \theta_2, 0\right).$$

This then gives the form of a general point in $Q \subset Q_{\text{free}}$. Now we note that such a point in Q is determined precisely by the matrices

$$\left(\begin{bmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{bmatrix}, \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 \\ \sin \theta_2 & \cos \theta_2 \end{bmatrix} \right) \in SO(2) \times SO(2).$$

Thus we take $Q = \mathbb{S}^1 \times \mathbb{S}^1$, making the identification made previously with $SO(2)$ and \mathbb{S}^1 . □

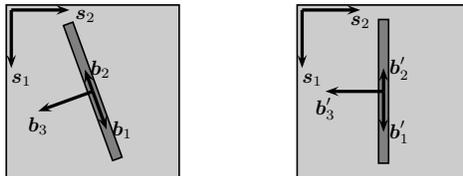
Example 2.15 (Rolling disk) Next we take the problem of a disk rolling on a plane P without slipping:



There is just a single rigid body, so we have $Q_{\text{free}} = SO(3) \times \mathbb{R}^3$. We choose frames as in the picture. Thus we say that

1. $O_{\text{spatial}} \in P$,
2. s_3 is orthogonal to P ,
3. O_{body} is at the center of mass of the disk, which we assume to be at the geometric center of the disk, and
4. b_3 points in the direction parallel to P , and orthogonal to direction of motion allowed by rolling.

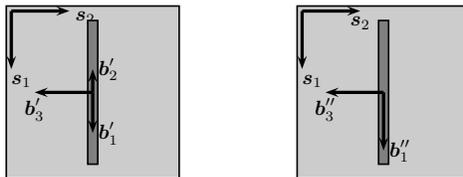
The position of O_{body} relative to O_{spatial} is given by a vector of the form (x, y, r) for $x, y \in \mathbb{R}$, and where $r > 0$ is the radius of the disk. To determine the matrix sending $\mathbf{R} \in \text{SO}(3)$ corresponding to the orientation of $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ relative to $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$, we proceed in three stages. We illustrate these three stages by sketching the bases as viewed by looking down on the plane P . The first stage produces $\{\mathbf{b}'_1, \mathbf{b}'_2, \mathbf{b}'_3\}$ by



Thus we rotate the frame $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ by an angle θ (see picture of disk for definition of θ) clockwise about the s_3 -axis. This corresponds to the orthogonal matrix

$$\mathbf{R}_1 = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

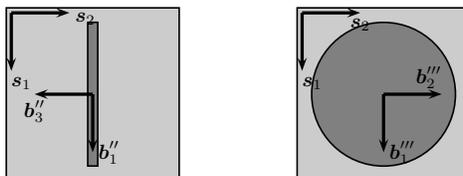
Next we apply a counterclockwise rotation about s_2 by the angle ϕ (again, see the figure for the definition of ϕ) to get $\{\mathbf{b}''_1, \mathbf{b}''_2, \mathbf{b}''_3\}$:



The matrix that does this is

$$\mathbf{R}_2 = \begin{bmatrix} \cos \phi & 0 & -\sin \phi \\ 0 & 1 & 0 \\ \sin \phi & 0 & \cos \phi \end{bmatrix}$$

Finally, to get to the basis $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ we rotate in the clockwise direction by $\frac{\pi}{2}$ above s_2 :



The matrix that does this is

$$\mathbf{R}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Note that we have $\mathbf{s}_i = \mathbf{R}_3 \mathbf{R}_2 \mathbf{R}_1 \mathbf{b}_i$, $i \in \{1, 2, 3\}$. Thus $\mathbf{b}_i = \mathbf{R}_1^T \mathbf{R}_2^T \mathbf{R}_3^T \mathbf{s}_i$, $i \in \{1, 2, 3\}$. If we note that we can take $\mathbf{s}_1 = (1, 0, 0)$, $\mathbf{s}_2 = (0, 1, 0)$, and $\mathbf{s}_3 = (0, 0, 1)$ (this amounts to choosing spatial coordinates adapted to our spatial frame), then we see that \mathbf{b}_i is the i th column of $\mathbf{R} \triangleq \mathbf{R}_1^T \mathbf{R}_2^T \mathbf{R}_3^T$. Thus \mathbf{R} as defined is that matrix in $\text{SO}(3)$ describing the orientation of the disk. A computation gives

$$\mathbf{R} = \begin{bmatrix} \cos \phi \cos \theta & \sin \phi \cos \theta & \sin \theta \\ \cos \phi \sin \theta & \sin \phi \sin \theta & -\cos \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix}.$$

To summarize, a typical point in $\text{SO}(3) \times \mathbb{R}^3$ corresponding to an admissible

configuration of the disk looks like

$$\left(\begin{bmatrix} \cos \phi \cos \theta & \sin \phi \cos \theta & \sin \theta \\ \cos \phi \sin \theta & \sin \phi \sin \theta & -\cos \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix}, (x, y, r) \right).$$

As with our other examples, it is easier to pull out the relevant part of \mathbf{Q} from \mathbf{Q}_{free} . To see how this might be done, note that the matrices \mathbf{R}_1 and \mathbf{R}_2 are exactly specified by the matrices

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}$$

in $\text{SO}(2)$, respectively. Thus the configuration space can be taken to be $\mathbf{Q} = \mathbb{R}^2 \times \text{SO}(2) \times \text{SO}(2)$, or, equivalently, $\mathbf{Q} = \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$. \square

- In the preceding three examples, we were quite systematic about determining the configuration manifold as a submanifold of \mathbf{Q}_{free} . In practice, after one has some experience with these sorts of things, one bypasses the steps we went through in the examples.
- Nonetheless, there are still occasions when even a seasoned veteran may want to

think somewhat carefully about what a configuration space is. For example, consider a rigid body fixed at a point in space by a universal joint. A universal joint does not allow the body to “twist.” If the universal joint is a ball joint, then the configuration manifold is simply $SO(3)$. The elimination of twist takes away one degree-of-freedom, so we expect Q to be two-dimensional for a universal joint. We have seen two possibilities of two-dimensional manifolds, $\mathbb{S}^1 \times \mathbb{S}^1$ and \mathbb{S}^2 . But neither of these is Q in this case. Indeed, one show that $Q = \mathbb{RP}^2$, which is “two-dimensional real projective space.”

Configuration spaces and differentiable manifolds (cont'd)

2.7. Choosing coordinates for a system

- Now that we have indicated how one in principal finds the configuration manifold Q for an interconnected mechanical system, we choose coordinates. This will be useful when we actually have to do computations.
- Note that since Q is a manifold, it possesses charts, so in some sense there is nothing to talk about. However, in most cases it is helpful to choose coordinate charts that mean something in terms of the physics.

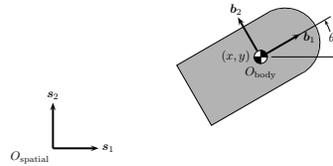
Example 2.16 (Planar rigid body (cont'd)) We return to our planar rigid body example, for which we had determined the configuration manifold to be $Q = \mathbb{S}^1 \times \mathbb{R}^2$. As a coordinate chart we choose (\mathcal{U}, ϕ) defined by

$$\mathcal{U} = Q \setminus \{((u, v), (x, y)) \in Q \mid u = -1, v = 0\},$$

$$\phi((u, v), (x, y)) = (\text{atan}(u, v), (x, y)).$$

We write these coordinates as (θ, x, y) , and we note that $\phi(\mathcal{U})$ as a subset of \mathbb{R}^3 consists of a “slab” that is infinite in x and y , and for which θ takes values from

$-\pi$ to π (boundaries not included). In terms of the physics, the coordinates (θ, x, y) are to be interpreted as follows:



□

Remarks 2.17 1. Note that it is quite unlikely that one would choose coordinates in any other way for this example. However, what we have done is recognize that this prescription of coordinates has meaning in terms of providing a chart for Q thought of as a manifold. In this sense, coordinates are given a quite precise meaning.

2. Note also that the coordinates do not actually cover all of Q . It will often be the case that one does choose a set of coordinates that does not cover the configuration manifold. There are two possible reasons why this might be:

(a) it may not be *possible* to cover Q with a single coordinate chart

(e.g., $Q = \mathbb{S}^1$);

(b) although it may be possible to cover Q with a single chart, perhaps the “natural” physical chart does not cover Q (this is actually the case with the planar body, although it is perhaps not obvious that it *is* possible to cover Q with a single chart for this example). □

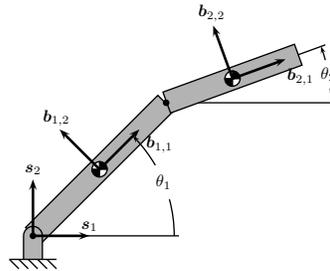
Example 2.18 (Two-link manipulator (cont’d)) Consider the two-link planar manipulator which has $Q = \mathbb{S}^1 \times \mathbb{S}^1$. Given that we have a natural way of defining coordinates for \mathbb{S}^1 , we just use this coordinate twice. Thus we define a chart (\mathcal{U}, ϕ) by

$$\mathcal{U} = Q \setminus (\{(x_1, y_1), (x_2, y_2) \mid x_1 = -1, y_1 = 0\}$$

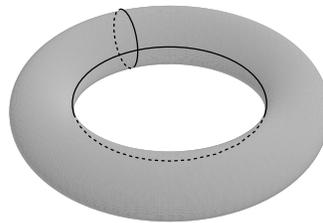
$$\cup \{(x_1, y_1), (x_2, y_2) \mid x_2 = -1, y_2 = 0\}),$$

$$\phi((x_1, y_1), (x_2, y_2)) = (\text{atan}(x_1, y_1), \text{atan}(x_2, y_2)).$$

Let us denote these coordinates by (θ_1, θ_2) . Physically, these coordinates are related to the configuration of the body as follows:

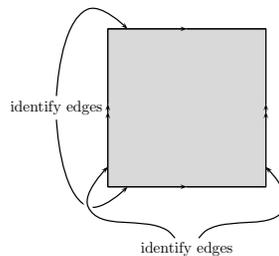


Let us try to get an idea of what Q “looks like.” First, let us define $\mathbb{T}^n = \mathbb{S}^1 \times \dots \times \mathbb{S}^1$, the n -fold Cartesian product, which we call the *n -torus*. Then Q is the 2-torus. One way of visualizing the 2-torus is as the surface of a “donut” in \mathbb{R}^3 :



The two circles indicate the portion of Q that we remove to define the coordinate

chart \mathcal{U} . Thus, in this representation of the 2-torus, the stuff that is removed is two circles. Another way to visualize Q in this case is to simply look at $\phi(\mathcal{U})$, which is the square $]-\pi, \pi[\times]-\pi, \pi[$: Note that the left and right edges of the square are really “the same,” since these boundaries correspond to one of the circles that was removed to define \mathcal{U} . In like manner, the top and bottom edges are “the same.” This can be represented by the following picture:



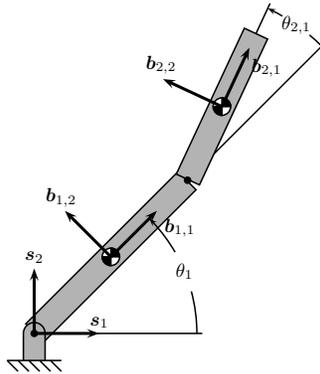
Thus one glues together the left and right edges, upon which one will have a cylinder, and then glues together the two boundaries of the cylinder, so producing the donut representation. □

Configuration spaces and differentiable manifolds (cont'd)

Example 2.19 (Two-link manipulator (cont'd)) Last time we made a choice of coordinates for $Q = \mathbb{S}^1 \times \mathbb{S}^1$, and we indicated how these coordinates relate to the physics of the system. Whereas for the planar body, there is perhaps a quite natural choice of coordinates, in this example, there is at least one other set of coordinates that makes sense. Let us define this as the chart (\mathcal{V}, ψ) given by

$$\begin{aligned} \mathcal{V} &= Q \setminus \{((x_1, y_1), (x_2, y_2)) \mid x_1 = -1, y_1 = 0\} \\ &\quad \cup \{((x_1, y_1), (x_2, y_2)) \mid x_2 = -x_1, y_2 = -y_1\} \\ \psi((x_1, y_1), (x_2, y_2)) &= (\text{atan}(x_1, y_1), \text{atan}(\cos(\text{atan}(x_1, y_1))x_2 + \sin(\text{atan}(x_1, y_1))y_2, \\ &\quad -\sin(\text{atan}(x_1, y_1))x_2 + \cos(\text{atan}(x_1, y_1))y_2)). \end{aligned}$$

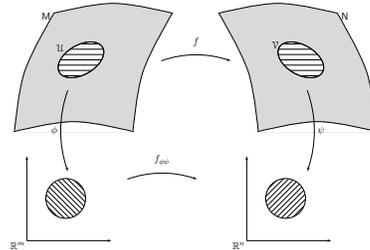
This coordinate chart looks complicated, but, if one sorts through the definition, one can see that the coordinates $(\theta_1, \theta_{2,1})$ in this chart are related to the position of the manipulator by:



2.8. Maps to and from manifolds

- Now that we have carefully constructed the configuration space of an interconnected mechanical system, let us talk briefly about things that can be done using the differentiable structure.
- One of the things one can do is talk about properties of maps between manifolds.

Definition 2.20 Let M and N be manifolds and let $f: M \rightarrow N$. Then f is *r -times continuously differentiable*, or of *class C^r* , if, for each $x \in M$, there exists charts (\mathcal{U}, ϕ) for M and (\mathcal{V}, ψ) for N with the following properties:



- (i) $x \in \mathcal{U}$,
- (ii) $f(\mathcal{U}) \subset \mathcal{V}$, and
- (iii) the map $f_{\phi\psi}$ from $\phi(\mathcal{U})$ to $\psi(\mathcal{V})$ defined by $f_{\phi\psi}(x) = \psi \circ f \circ \phi^{-1}(x)$ is r -times continuously differentiable. □

- The map $f_{\phi\psi}$ in the definition is the *local representative* of f .
- *Idea*: To talk about differentiability of a map between manifolds, one simply chooses coordinates, represents the map in one's coordinates, and then talks about differentiability of the coordinate representation, which makes sense.

- We denote the class C^r maps from M to N by $C^r(M; N)$.
- Two sorts of maps will be of particular interest.
 1. $N = \mathbb{R}$: Maps from a manifold to \mathbb{R} are called *functions*, and $C^r(M)$ denotes the class C^r functions from M to \mathbb{R} .
 2. $M = \mathbb{R}$: Maps from \mathbb{R} to a manifold N are called *curves*. A typical curve will be denoted by γ .

Example 2.21 We take $M = \mathbb{S}^2$ and define $f: \mathbb{S}^2 \rightarrow \mathbb{R}$ by $f(x, y, z) = z$. We claim that f is a C^∞ -function. We can check this by applying the definition. Thus we choose a coordinate chart, find its local representative, and check that the local representative is C^∞ . Let us choose, for example, longitude/latitude coordinates discussed in Example 2.8. It is then easy to see that the local representative of f is $(\theta, \phi) \mapsto \cos \phi$. This function is clearly infinitely differentiable. To actually show that f is infinitely differentiable, one should choose another chart covering the points not covered by latitude/longitude coordinates, and make sure the local representative of f is also infinitely differentiable. This can be done. □

- An easier way to check the differentiability of a function like that in the preceding example is to use the following result, which follows directly from the definitions of submanifold, and class C^r map.

Proposition 2.22 *Let M and N be manifolds, let S be a submanifold of M , and let $f \in C^r(M; N)$. Then $f|_S \in C^r(S; N)$.*

- *Idea:* To check the differentiability of a function defined on a submanifold, it suffices to check differentiability on the bigger space, which is sometimes easier.

Example 2.23 We continue with our previous example where $M = \mathbb{S}^2$ is a submanifold of \mathbb{R}^3 , and $f(x, y, z) = z$. Clearly, f is infinitely differentiable on \mathbb{R}^3 . The result above says that this immediately implies that f is also infinitely differentiable when restricted to \mathbb{S}^2 , without having to laboriously check local representatives, etc. □

Configuration spaces and differentiable manifolds (cont'd)

Example 2.24 Next let us consider an example of a curve. We take $M = \mathbb{S}^2$ again, and define a map $\gamma: \mathbb{R} \rightarrow \mathbb{S}^2$ by $\gamma(t) = (\cos t, \sin t, 0)$. We claim that γ is an infinitely differentiable curve. To verify this by hand, we need to show that the local representative for γ in a chart is infinitely differentiable. We again take longitude/latitude coordinates (θ, ϕ) , and in these coordinates the local representative of γ is $t \mapsto (t, \frac{\pi}{2})$. One should be careful to take into account the domain of validity of longitude/latitude coordinates. After doing so, one sees that this map is only defined when t is not an odd multiple of π . However, in the range of validity, we see that the local representative is infinitely differentiable. To show that γ is infinitely differentiable, one should choose another chart that covers the remaining part of \mathbb{S}^2 , and check that the local representative here is also infinitely differentiable. This is possible. □

- As with checking the differentiability of functions defined on submanifolds, there is an easy way to check the differentiability of a curve, or more generally, a map, taking values in a submanifold.

Proposition 2.25 *Let M and N be manifolds, let $S \subset M$ be a submanifold, and let $f: N \rightarrow M$ be a class C^r map having the property that $\text{image}(f) \subset S$. Then f is a class C^r map from N into S .*

- *Idea:* To check the differentiability of a curve taking values in a submanifold, it suffices to check the differentiability of the curve taking values in the bigger space, which is sometimes easier.

Example 2.26 For our previous example, \mathbb{S}^2 is a submanifold of \mathbb{R}^3 , and since the curve $t \mapsto (\cos t, \sin t, 0)$ is clearly an infinitely differentiable curve in \mathbb{R}^3 , it is also an infinitely differentiable curve in \mathbb{S}^2 . \square

- Exercises 2.21 and 2.24 might give one the impression that to check the differentiability of a curve or function, it suffices to use a chart that covers “most” of the manifold. This is false, as the following example shows.

Example 2.27 We take $M = \mathbb{S}^1$, and define a function on M by $f(x, y) = \text{atan}(x, y)$. Let us choose a chart (\mathcal{U}, ϕ) for \mathbb{S}^1 that covers all but one point:

$$\mathcal{U} = \mathbb{S}^1 \setminus \{(-1, 0)\}, \quad \phi(x, y) = \text{atan}(x, y).$$

Let us denote the coordinate in this case by θ . In this chart, the local representative of f is $\theta \mapsto \theta$. This is certainly an infinitely differentiable function. However, to check the infinite differentiability of f , we need to choose another chart that covers the point $(-1, 0)$. Let us define a chart (\mathcal{V}, ψ) by

$$\mathcal{V} = \mathbb{S}^1 \setminus \{(1, 0)\}, \quad \psi(x, y) = \text{atan}(-x, -y),$$

and denote the coordinate in this chart as $\tilde{\theta}$. The local representative of f in this chart is then given by

$$\tilde{\theta} \mapsto \begin{cases} -\pi + \tilde{\theta}, & \tilde{\theta} > 0, \\ \pi + \tilde{\theta}, & \tilde{\theta} \leq 0. \end{cases}$$

Note that this local representative is not even continuous! This shows that it is *not* sufficient to check differentiability of a map in a single chart that covers “most” of the manifold. \square

2.9. Comments on smoothness of maps

- In the text, we are fairly careful to specify the smoothness of objects. However, in this course, it very often suffices to simply assume that everything is infinitely differentiable, unless otherwise stated.
- Indeed, the only objects in this course that will *not* be assumed to be infinitely differentiable are controls, when we get around to talking about them.
- At times in the text, reference will be made to objects that are C^ω , or equivalently, analytic. This does not mean analytic in the sense of complex variables. In the context of this course, *analytic*, or more exactly, *real analytic*, means that the Taylor series for an object converges to the object. Of course, this definition means one works in a coordinate chart.

2.10. The forward kinematic maps

- We next introduce a very simple object, one which we have in actuality already used.

Definition 2.28 Consider an interconnected mechanical system with k interconnected rigid bodies and configuration manifold $Q \subset Q_{\text{free}}$. For $a \in \{1, \dots, k\}$, the *ath forward kinematic map* is the map $\Pi_a: Q \rightarrow \text{SO}(3) \times \mathbb{R}^3$ which assigns to a point in Q the position of body a corresponding to that configuration. \square

- As we say, we have really seen the forward kinematic map already. Let us illustrate that this is indeed the case by looking at our examples.

Example 2.29 (Planar rigid body (cont'd)) Recall that $Q = \mathbb{S}^1 \times \mathbb{R}^2$. There is one body, and the forward kinematic map $\Pi_1: Q \rightarrow \text{SO}(3) \times \mathbb{R}^3$ is given in coordinates by

$$\Pi_1(\theta, x, y) = \left(\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, (x, y, 0) \right).$$

Note that the forward kinematic map is already contained in our explicit description of Q as a submanifold of Q_{free} . \square

Example 2.30 (Two link manipulator (cont'd)) Here we had $Q = \mathbb{S}^1 \times \mathbb{S}^1$. There are two bodies and the coordinate representations of the forward kinematic maps are

$$\Pi_1(\theta_1, \theta_2) = \left(\begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 \\ \sin \theta_1 & \cos \theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\frac{\ell_1}{2} \cos \theta_1, \frac{\ell_1}{2} \sin \theta_1, 0 \right) \right),$$

$$\Pi_2(\theta_1, \theta_2) = \left(\begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & 0 \\ \sin \theta_2 & \cos \theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\ell_1 \cos \theta_1 + \frac{\ell_2}{2} \cos \theta_2, \ell_2 \sin \theta_2 + \frac{\ell_2}{2} \sin \theta_2, 0 \right) \right).$$

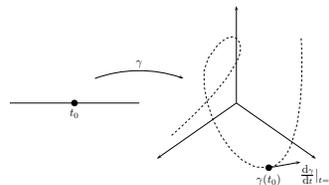
Again, this is a mere matter of giving a name to something we have already seen. \square

3. Velocity and the tangent bundle

- We now talk about something different. We use the physical notion of velocity to motivate the introduction of a new object: the tangent bundle.

3.1. Velocity

- We first need to understand well what “velocity” is. To come to grips with this, let us consider a particle moving in \mathbb{R}^3 . The particle's motion is described by a curve $\gamma: \mathbb{R} \rightarrow \mathbb{R}^3$. Thus the position of the particle at time t_0 is $\gamma(t_0) \in \mathbb{R}^3$ and the velocity of the particle at time t_0 is $\left. \frac{d\gamma}{dt} \right|_{t=t_0} \in \mathbb{R}^3$.
- The way to imagine this so that it will be useful in our subsequent discussion is as follows:



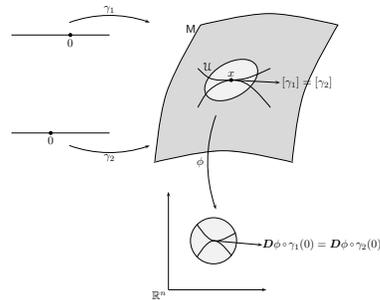
- For the curve in \mathbb{R}^3 , the tangent vector lived also in \mathbb{R}^3 , and we imagined the origin for the set of velocities to be at the point on the curve whose velocity was being computed. For a manifold, where “velocity” should live is not so clear.

Definition 3.1 Let M be a manifold and let $x \in M$.

- (i) A **curve at x** is a C^1 -curve $\gamma: I \rightarrow M$ with the property that $0 \in \text{int}(I)$ and $\gamma(0) = x$.
- (ii) Two curves at x , γ_1 and γ_2 , are **equivalent** if, for a chart (U, ϕ) around x , it holds that $D(\phi \circ \gamma_1)(0) = D(\phi \circ \gamma_2)(0)$.
- (iii) A **tangent vector** at x is an equivalence class of curves under the above equivalence relation. The set of tangent vectors at x is denoted $T_x M$ and is called the **tangent space** at x .
- (iv) The **tangent bundle** to M is the collection of all tangent spaces:

$$TM = \bigcup_{x \in M} T_x M. \quad \square$$

- **Idea:** One represents the two curves in a set of coordinates, and asks that they have the same velocity in the coordinates. For this to make sense, one should, of course, check that this idea is independent of choice of coordinates. This is easily done.
- **Idea:** For an interconnected mechanical system with configuration manifold Q , points in Q are **positions** of the system, points in $T_q Q$ are velocities at position q , and TQ is the collection of all velocities at all possible positions.
- It is important to note that, in this way of thinking of things, velocity does not exist independent of position.
- We would like to have a way of expressing velocities in coordinates.



3.2. Coordinate representations for velocity

- Let (\mathcal{U}, ϕ) be a chart for M . We wish to represent points in $T_x M$ for $x \in \mathcal{U}$.
- Note that points in \mathcal{U} are represented by elements $\phi(x) \in \phi(\mathcal{U}) \subset \mathbb{R}^n$. Now let $x \in \mathcal{U}$ and let γ be a curve at x which therefore defines a tangent vector $[\gamma] \in T_x M$. In the chart, $[\gamma]$ is prescribed by $D(\phi \circ \gamma)(0) \in \mathbb{R}^n$.
- *Punchline:* The coordinate representation of a tangent vector at $x \in \mathcal{U}$ is specified by $(\phi(x), D(\phi \circ \gamma)(0)) \in \phi(\mathcal{U}) \times \mathbb{R}^n$.
- Note that, if we write $(x^1(t), \dots, x^n(t)) = \phi \circ \gamma(t)$, then

$$D(\phi \circ \gamma)(0) = (\dot{x}^1(0), \dots, \dot{x}^n(0)).$$

- We shall adopt the notation

$$\underbrace{((x^1, \dots, x^n))}_{\text{in } \phi(\mathcal{U})}, \underbrace{(v^1, \dots, v^n)}_{\text{in } \mathbb{R}^n}$$

to denote the coordinate representation of a typical tangent vector at a point in \mathcal{U} .

- *Important question:* How are two different coordinate representations of the same tangent vector related?
- To answer this question, let $x \in M$ and let (\mathcal{U}, ϕ) and $(\tilde{\mathcal{U}}, \tilde{\phi})$ be charts around x . Let $[\gamma] \in T_x M$ and let $D(\phi \circ \gamma)(0)$ and $D(\tilde{\phi} \circ \gamma)(0)$ be the coordinate representations of $[\gamma]$.
- It is convenient to write coordinates for (\mathcal{U}, ϕ) as (x^1, \dots, x^n) and coordinates for $(\tilde{\mathcal{U}}, \tilde{\phi})$ as $(\tilde{x}^1, \dots, \tilde{x}^n)$. With this notation, we write

$$\phi \circ \gamma(t) = (x^1(t), \dots, x^n(t)),$$

$$\tilde{\phi} \circ \gamma(t) = (\tilde{x}^1(t), \dots, \tilde{x}^n(t)).$$

- Then, using the Chain Rule,

$$\dot{\tilde{x}}^i(0) = \sum_{j=1}^n \frac{\partial \tilde{x}^i}{\partial x^j}(\tilde{x}(0)) \dot{x}^j(0), \quad i \in \{1, \dots, n\}. \quad (1)$$

This is *exactly* what velocities should do when we change coordinates!

- An alternate way to write (1) is

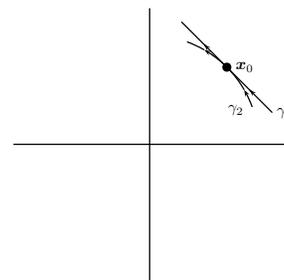
$$\underbrace{D(\tilde{\phi} \circ \gamma)(0)}_{\simeq \dot{\tilde{x}}^i(0)} = \underbrace{D(\tilde{\phi} \circ \phi^{-1})(\phi \circ \gamma(0))}_{\simeq \frac{\partial \tilde{x}^i}{\partial x^j}(\mathbf{x}(0))} \underbrace{D(\phi \circ \gamma)(0)}_{\simeq \dot{x}^j(0)}. \quad (2)$$

- That is to say, the velocity in one set of coordinates is obtained by multiplying the velocity in the other set of coordinates by the Jacobian of the overlap map. If one were to think about it, this probably is in keeping with some parts of your past experience.
- Note that (1) and (2) are really the same equation. The representation (1) is more transparent notationally. The representation (2) is more compact, and makes explicit the connection with the overlap map introduced in Definition 2.2.

Velocity and the tangent bundle (cont'd)

Example 3.2 Let us look at something simple; our manifold $M = \mathbb{R}^2$ with Cartesian coordinates (x, y) and polar coordinates (r, θ) . The charts were denoted (\mathcal{U}_1, ϕ_1) and (\mathcal{U}_2, ϕ_2) , respectively. Let us consider the representation of a specific tangent vector in each of these two coordinate charts. Let

$\mathbf{x}_0 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \in M$. Consider the curves at \mathbf{x}_0 given by $\gamma_1(t) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) + t(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $\gamma_2(t) = (\cos(t + \frac{\pi}{4}), \sin(t + \frac{\pi}{4}))$. We claim that these curves are equivalent at \mathbf{x}_0 . To see that this is the case, we need to show that, in some (and therefore any) coordinate chart, the local representatives of the two curves have equal derivatives at $t = 0$. Let us do this with the more difficult of the two coordinate charts, just for fun.



The local representatives the curves in polar coordinates are

$$\phi_2 \circ \gamma_1(t) = \left(\sqrt{\left(\frac{1}{\sqrt{2}}(1-t)\right)^2 + \left(\frac{1}{\sqrt{2}}(1+t)\right)^2}, \text{atan}(1-t, 1+t) \right), \quad \phi_2 \circ \gamma_2(t) = \left(1, \frac{\pi}{4} + t \right).$$

The derivatives of these curves at $t = 0$ are

$$D(\phi_2 \circ \gamma_1)(0) = (0, 1), \quad D(\phi_2 \circ \gamma_2)(0) = (0, 1).$$

Thus the two curves are indeed equivalent, and so define the same tangent vector.

Let us show how the two coordinate representations of this tangent vector are related. We have already shown that the tangent vector in polar coordinates is represented by

$$((r, \theta), (v_r, v_\theta)) = \left(\left(1, \frac{\pi}{4} \right), (0, 1) \right).$$

In Cartesian coordinates we compute

$$D(\phi_1 \circ \gamma_1)(0) = D(\phi_1 \circ \gamma_2)(0) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right).$$

(Note that the first equality is guaranteed since the curves are equivalent at x_0 .)

Thus we represent the tangent vector in Cartesian coordinates by

$$((x, y), (v_x, v_y)) = \left(\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right).$$

The rule for how these vectors should be related is

$$v_r = \frac{\partial r}{\partial x} v_x + \frac{\partial r}{\partial y} v_y = \frac{x}{\sqrt{x^2 + y^2}} v_x + \frac{y}{\sqrt{x^2 + y^2}} v_y,$$

$$v_\theta = \frac{\partial \theta}{\partial x} v_x + \frac{\partial \theta}{\partial y} v_y = -\frac{y}{x^2 + y^2} v_x + \frac{x}{x^2 + y^2} v_y.$$

Substituting in the values for $((x, y), (v_x, v_y))$ we get $(v_r, v_\theta) = (0, 1)$, as desired. \square

- Notice that if (\mathcal{U}, ϕ) is a chart for M , the map

$$T_x M \ni [\gamma] \mapsto (\phi(x), D(\phi \circ \gamma)(0)) \in \phi(\mathcal{U}) \times \mathbb{R}^n$$

defines a coordinate chart for $T\mathcal{U} = \cup_{x \in \mathcal{U}} T_x M$, denoted by $(T\mathcal{U}, T\phi)$.

- One additionally sees, by virtue of our previous machinations, that if (\mathcal{U}_1, ϕ_1) and (\mathcal{U}_2, ϕ_2) are overlapping charts for M , then the overlap map for the corresponding tangent bundle charts is

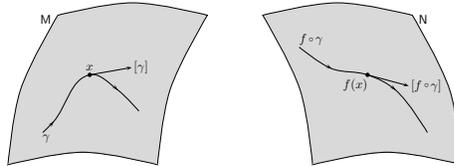
$$\phi_1(\mathcal{U}_1 \cap \mathcal{U}_2) \times \mathbb{R}^n \ni (\mathbf{x}, \mathbf{v}) \mapsto (\phi_{12}(\mathbf{x}), D\phi_{12}(\mathbf{x}) \cdot \mathbf{v}) \in \phi_2(\mathcal{U}_1 \cap \mathcal{U}_2) \times \mathbb{R}^n.$$

This is yet another manifestation of the transformation rule for velocities.

Moreover, it gives us...

- **Punchline:** TM is a manifold, since if $\mathcal{A} = \{(\mathcal{U}_a, \phi_a)\}_{a \in A}$ is an atlas for M, then $T\mathcal{A} \triangleq \{(T\mathcal{U}_a, T\phi_a)\}_{a \in A}$ is an atlas for TM. The charts in the atlas $T\mathcal{A}$ are called **natural charts**.

- Let $f: M \rightarrow N$ be an infinitely differentiable map, let $x \in M$, and let γ be a curve at x . Then $f \circ \gamma$ is a curve at $f(x)$.



- We then define a map $T_x f: T_x M \rightarrow T_{f(x)} N$ by $T_x f([\gamma]) = [f \circ \gamma]$. Doing this for each $x \in M$ defines a map $Tf: TM \rightarrow TN$, which is the **tangent map** of f .

Velocity and the tangent bundle (cont'd)

- Let us write the local representative of Tf in a natural chart. Let us denote natural coordinates for TM by $((x^1, \dots, x^m), (v^1, \dots, v^m))$, and let us write the local representative for f as

$$(x^1, \dots, x^m) \mapsto (f^1(\mathbf{x}), \dots, f^n(\mathbf{x})).$$

The local representative of γ we write as

$$t \mapsto (x^1(t), \dots, x^m(t)),$$

so that the local representative of $f \circ \gamma$ is

$$t \mapsto (f^1(\mathbf{x}(t)), \dots, f^n(\mathbf{x}(t))).$$

Therefore, the coordinate representation for $[\gamma]$ is

$$((x^1(0), \dots, x^m(0)), (\dot{x}^1(0), \dots, \dot{x}^m(0)))$$

and the coordinate representation for $[f \circ \gamma]$ is

$$\left((f^1(\mathbf{x}(0)), \dots, f^n(\mathbf{x}(0))), \left(\sum_{j=1}^m \frac{\partial f^1}{\partial x^j}(\mathbf{x}(0)) \dot{x}^j(0), \dots, \sum_{j=1}^m \frac{\partial f^n}{\partial x^j}(\mathbf{x}(0)) \dot{x}^j(0) \right) \right).$$

- More compactly, the local representative of Tf is

$$\phi(\mathcal{U}) \times \mathbb{R}^m \ni (\mathbf{x}, \mathbf{v}) \mapsto (f_{\phi\psi}(\mathbf{x}), Df_{\phi\psi}(\mathbf{x}) \cdot \mathbf{v}) \in \psi(\mathcal{V}) \times \mathbb{R}^n,$$

where (\mathcal{U}, ϕ) is the chart on M and (\mathcal{V}, ψ) is the chart on N .

- **Punchline:** The local representative of Tf is defined by the derivative (i.e., Jacobian) of the local representative of f . Thus one should think of Tf as being the derivative of f .

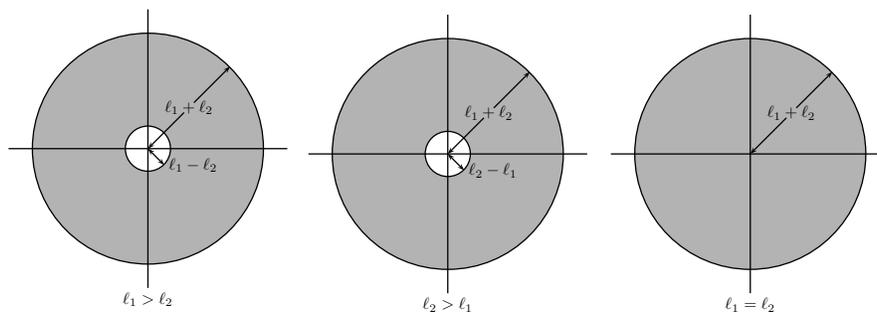
Example 3.3 (Planar two-link manipulator) To get some intuition about maps between manifolds, and about derivatives of such maps, we consider the two-link planar manipulator that has been used previously. The configuration manifold is $Q = \mathbb{S}^1 \times \mathbb{S}^1$. We define a map $f: Q \rightarrow \mathbb{R}^2$ by asking that $f(q)$ be the position in the plane of the tip of the second link. This would be the position of the end effector of the robot.

Let us ask some basic questions about this map.

1. Is f surjective? If not, describe $\text{image}(f)$.
2. Is f injective? If not, describe the character of $f^{-1}(x, y)$ for $(x, y) \in \text{image}(f)$.

It is quite clear that f is not surjective, since points that are distance greater than

$\ell_1 + \ell_2$ (ℓ_i is the length of link i) from the base of the robot are not reachable using f . A more detailed description is given by the following picture:



Thus the image of f is an annulus when $\ell_1 \neq \ell_2$, and is a disk when $\ell_1 = \ell_2$.

What about injectivity? A little thought gives the following characterization of the sets $f^{-1}(x, y)$.

	$\ell_1 > \ell_2$	$\ell_2 > \ell_1$	$\ell_1 = \ell_2$
$(x, y) \in \text{bd}(\text{image}(f))$	single point	single point	single point
$(x, y) \in \text{int}(\text{image}(f))$	two points	two points	two points
$(x, y) = (0, 0)$	n/a	n/a	copy of \mathbb{S}^1

The entries in the table give the character of $f^{-1}(x, y)$

The idea is that when one is in the interior of $\text{image}(f)$, there are two possible robot configurations giving the same position of the tip of the second link. In the case where we are at the boundary of $\text{image}(f)$, there is only one possible robot configuration giving that position of the tip of the second link. A degenerate case arise when $\ell_1 = \ell_2$ and $(x, y) = (0, 0)$. In this case, one can rotate the robot links together, and the tip of the second link will not move.

Now let us look at the derivative Tf of the map f . Rather than just compute it, let us think about what the derivative means. If one is at configuration $q \in Q$ moving with velocity $v_q \in T_q Q$, then $Tf(v_q)$ is the velocity of the tip of the second link.

An interesting question is the following. Are there configurations q and nonzero velocities $v_q \in T_q Q$ for which $Tf(v_q) = 0$. This means that even though the links may be moving, it is possible that the tip of link two is not moving. Rephrasing the

question mathematically, are there configurations q for which $\ker(T_q f)$ is nontrivial (i.e., consists of more than the zero vector)? To answer this question, one can compute Tf . Note that, in the coordinates (θ_1, θ_2) , the local representative of f is

$$(\theta_1, \theta_2) \mapsto (\ell_1 \cos \theta_1 + \ell_2 \cos \theta_2, \ell_1 \sin \theta_1 + \ell_2 \sin \theta_2).$$

The local representative of Tf is then

$$\begin{aligned} &((\theta_1, \theta_2), (v_1, v_2)) \\ &\mapsto ((\ell_1 \cos \theta_1 + \ell_2 \cos \theta_2, \ell_1 \sin \theta_1 + \ell_2 \sin \theta_2), (J_{11}v_1 + J_{12}v_2, J_{21}v_1 + J_{22}v_2)), \end{aligned}$$

where J_{ij} , $i, j = \{1, 2\}$, are the components of the Jacobian of the local representative of f :

$$\begin{bmatrix} -\ell_1 \sin \theta_1 & -\ell_2 \sin \theta_2 \\ \ell_1 \cos \theta_1 & \ell_2 \cos \theta_2 \end{bmatrix}$$

One can then readily verify that $T_q f$ has nontrivial kernel if and only if $f(q) \in \text{int}(\text{image}(f))$, provided that $\ell_1 \neq \ell_2$. When $\ell_1 = \ell_2$, then $T_q f$ has nontrivial kernel if and only if (1) $f(q) \in \text{int}(\text{image}(f))$ and (2) $f(q) \neq (0, 0)$.

Moral of the story: There is a relation ship between the derivative changing rank, and changes in the character of $f^{-1}(x, y)$. This is generally true. However, the main

idea of working out this example in detail is to try to get some intuition for the behavior of maps between manifolds. \square

- We next introduce some convenient notation. Let $\gamma: \mathbb{R} \rightarrow M$ be a curve. Then $T\gamma$ is a map from $T\mathbb{R}$ to TM .
- Since \mathbb{R} possesses a natural global coordinate chart, $T\mathbb{R} \simeq \mathbb{R} \times \mathbb{R}$. We then define $\gamma'(t) = T\gamma \cdot 1$.
- If the local representative for γ is

$$t \mapsto (x^1(t), \dots, x^n(t)),$$

then the local representative of γ' in natural coordinates for the tangent bundle is

$$t \mapsto ((x^1(t), \dots, x^n(t)), (\dot{x}^1(t), \dots, \dot{x}^n(t))).$$

- The picture one should have in mind for γ' is that $\gamma'(t)$ is the velocity of the curve γ at time t :



- Finally, some convenient terminology I forgot to mention earlier. While it is sometimes convenient to write tangent vectors as $[\gamma]$ to emphasize their definition as equivalence classes of curves, it is sometimes cumbersome to do this. Thus we shall frequently write a point in the tangent space $T_x M$ as v_x . The subscript “ x ” reminds us at what base point this tangent vector sits.

4. Angular velocity

4.1. Velocity of rigid body motion

- Suppose that a body is undergoing a motion described by $t \mapsto (\mathbf{R}(t), \mathbf{r}(t)) \in SO(3) \times \mathbb{R}^3$ (look back to our discussion of the configuration of a rigid body, if you forget why this describes the motion of a rigid body).
- We shall suppose that $\mathbf{r}(t)$ describes the position of the center of mass, corresponding to our choice of body origin at the center of mass. Then $\dot{\mathbf{r}}(t)$ is the *translational velocity*.
- The description of angular velocity is less straightforward, and we just provide the “answer.”
- First define matrices $\widehat{\boldsymbol{\omega}}(t) = \dot{\mathbf{R}}(t)\mathbf{R}^T(t)$ and $\widehat{\boldsymbol{\Omega}}(t) = \mathbf{R}^T(t)\dot{\mathbf{R}}(t)$.

Lemma 4.1 *The matrices $\widehat{\boldsymbol{\omega}}(t)$ and $\widehat{\boldsymbol{\Omega}}(t)$ are skew-symmetric.*

Proof: We do this for $\widehat{\boldsymbol{\Omega}}(t)$, leaving $\widehat{\boldsymbol{\omega}}(t)$ as an exercise. Differentiating the equality $\mathbf{R}^T(t)\mathbf{R}(t) = \mathbf{I}_3$ with respect to t gives

$$\dot{\mathbf{R}}^T(t)\mathbf{R}(t) + \mathbf{R}^T(t)\dot{\mathbf{R}}(t) = \mathbf{0}.$$

Therefore,

$$(\mathbf{R}^T(t)\dot{\mathbf{R}}(t))^T = -\mathbf{R}^T(t)\dot{\mathbf{R}}(t),$$

as desired. ■

- Let us write

$$\widehat{\boldsymbol{\omega}}(t) = \begin{bmatrix} 0 & -\omega^3(t) & \omega^2(t) \\ \omega^3(t) & 0 & -\omega^1(t) \\ -\omega^2(t) & \omega^1(t) & 0 \end{bmatrix}, \quad \widehat{\boldsymbol{\Omega}}(t) = \begin{bmatrix} 0 & -\Omega^3(t) & \Omega^2(t) \\ \Omega^3(t) & 0 & -\Omega^1(t) \\ -\Omega^2(t) & \Omega^1(t) & 0 \end{bmatrix}.$$

- This allows us to define $\boldsymbol{\omega}(t) = (\omega^1(t), \omega^2(t), \omega^3(t))$ and $\boldsymbol{\Omega}(t) = (\Omega^1(t), \Omega^2(t), \Omega^3(t))$, which are the *spatial angular velocity* and *body angular velocity* for the motion.

- Note that we have implicitly define a map $\hat{\cdot}: \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ from \mathbb{R}^3 into the set $\mathfrak{so}(3)$ of skew-symmetric 3×3 matrices. It is easy to verify that this is an isomorphism of vector spaces, and satisfies (and can indeed be defined by) $\hat{a}b = a \times b$, where $a, b \in \mathbb{R}^3$ and where \times denotes the vector cross-product. We denote the inverse of the “hat map” by $\check{\cdot}: \mathfrak{so}(3) \rightarrow \mathbb{R}^3$.
- *Physical interpretations:* At a given instant in time, the body’s rotational motion in space is about the axis $\omega(t) \in \mathbb{R}^3$, and the magnitude of the angular velocity is $\|\omega(t)\|_{\mathbb{R}^3}$. The body angular velocity is how an observer fixed in the body would measure the vector $\omega(t) \in \mathbb{R}^3$ in the body frame.

4.2. Velocity for interconnected mechanical systems

- Now suppose that we have an interconnected mechanical system with k rigid bodies and with configuration manifold $Q \subset Q_{\text{free}}$. Let $\Pi_a: Q \rightarrow \text{SO}(3) \times \mathbb{R}^3$, $a \in \{1, \dots, k\}$, denote the forward kinematic maps.
- Define maps $\rho_{\text{spatial}}, \rho_{\text{body}}, \lambda_{\text{spatial}}, \lambda_{\text{body}}: T(\text{SO}(3) \times \mathbb{R}^3) \rightarrow \mathbb{R}^3$ by

$$\rho_{\text{spatial}}(\mathbf{R}, \mathbf{r}, \mathbf{A}, \mathbf{v}) = (\mathbf{A}\mathbf{R}^T)^\vee,$$

$$\rho_{\text{body}}(\mathbf{R}, \mathbf{r}, \mathbf{A}, \mathbf{v}) = (\mathbf{R}^T\mathbf{A})^\vee,$$

$$\lambda_{\text{spatial}}(\mathbf{R}, \mathbf{r}, \mathbf{A}, \mathbf{v}) = \mathbf{v},$$

$$\lambda_{\text{body}}(\mathbf{R}, \mathbf{r}, \mathbf{A}, \mathbf{v}) = \mathbf{v}.$$

(Note that in writing a point in $T(\text{SO}(3) \times \mathbb{R}^3)$ we are using two facts:

1. $\text{SO}(3) \times \mathbb{R}^3$ is a submanifold of $\mathbb{R}^{3 \times 3} \times \mathbb{R}^3$;
2. $T(\mathbb{R}^{3 \times 3} \times \mathbb{R}^3) \simeq (\mathbb{R}^{3 \times 3} \times \mathbb{R}^3) \times (\mathbb{R}^{3 \times 3} \times \mathbb{R}^3)$.

You should probably think a little about this.)

- Now suppose that the system is undergoing a motion $t \mapsto \gamma(t) \in \mathbb{Q}$. Then body a is undergoing the physical motion $t \mapsto \Pi_a \circ \gamma(t) \in \text{SO}(3) \times \mathbb{R}^3$. We can then define the body and spatial velocities of body a just as we did for a single body above.
- Explicitly, the spatial angular velocity of body a is

$$\rho_{\text{spatial}} \left(\left. \frac{d}{dt} \right|_{t=0} \Pi_a \circ \gamma(t) \right),$$

the body angular velocity of body a is

$$\rho_{\text{body}} \left(\left. \frac{d}{dt} \right|_{t=0} \Pi_a \circ \gamma(t) \right),$$

the spatial translational velocity of body a is

$$\lambda_{\text{spatial}} \left(\left. \frac{d}{dt} \right|_{t=0} \Pi_a \circ \gamma(t) \right),$$

and body translational velocity of body a is

$$\lambda_{\text{body}} \left(\left. \frac{d}{dt} \right|_{t=0} \Pi_a \circ \gamma(t) \right).$$

Example 4.2 (Planar rigid body (cont'd)) This is all easily illustrated with an example. We consider the planar rigid body with configuration manifold $\mathbb{Q} = \mathbb{S}^1 \times \mathbb{R}^2$. The key thing is the forward kinematic map:

$$\Pi_1(\theta, x, y) = \left(\underbrace{\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_1}, \underbrace{(x, y, 0)}_{r_1} \right).$$

To compute the spatial angular velocity we compute

$$\dot{R}_1 R_1^T = \begin{bmatrix} -\sin \theta \dot{\theta} & -\cos \theta \dot{\theta} & 0 \\ \cos \theta \dot{\theta} & -\sin \theta \dot{\theta} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \widehat{(0, 0, \dot{\theta})}.$$

This gives the coordinate expression

$$\rho_{\text{spatial}} \left(\left. \frac{d}{dt} \right|_{t=0} \Pi_1 \circ \gamma(t) \right) = (0, 0, \dot{\theta}).$$

In similar manner we compute

$$\begin{aligned}\rho_{\text{body}}\left(\frac{d}{dt}\Big|_{t=0} \Pi_1 \circ \gamma(t)\right) &= (0, 0, \dot{\theta}), \\ \lambda_{\text{spatial}}\left(\frac{d}{dt}\Big|_{t=0} \Pi_a \circ \gamma(t)\right) &= (\dot{x}, \dot{y}, 0), \\ \lambda_{\text{body}}\left(\frac{d}{dt}\Big|_{t=0} \Pi_a \circ \gamma(t)\right) &= (\dot{x}, \dot{y}, 0).\end{aligned}$$

This begins to show the value of the forward kinematic maps, since they reduce this computation to one that is merely rote. \square

5. Vector fields

Definition 5.1 A *vector field* on a manifold M is a class C^∞ map $X: M \rightarrow TM$ with the property that $X(x) \in T_x M$. \square

- *Idea:* A vector field assigns a tangent vector to each point in M .
- To represent a vector field in coordinates is a simple matter. In coordinates (x^1, \dots, x^n) the local representative of X is

$$(x^1, \dots, x^n) \mapsto ((x^1, \dots, x^n), (X^1(x), \dots, X^n(x)))$$

for some functions X^1, \dots, X^n of the coordinates.

- The functions X^1, \dots, X^n are called the *components* of X in the coordinates (x^1, \dots, x^n) .
- Vector fields have many facets, and we explore some of them.

5.1. Vector fields and differential equations

Definition 5.2 An *integral curve* of a vector field X at $x \in M$ is a curve γ at x having the property that $\gamma'(t) = X(\gamma(t))$ for all times t for which γ is defined. \square

- Let us understand what an integral curve is by writing the defining equality $\gamma'(t) = X(\gamma(t))$ in coordinates.

- The coordinate representation for $\gamma'(t)$ is

$$((x^1(t), \dots, x^n(t)), (\dot{x}^1(t), \dots, \dot{x}^n(t))).$$

- The coordinate representative for $X(\gamma(t))$ is

$$((x^1(t), \dots, x^n(t)), (X^1(x(t)), \dots, X^n(x(t)))).$$

- The equality is then

$$\dot{x}^1(t) = X^1(x(t))$$

$$\vdots$$

$$\dot{x}^n(t) = X^n(x(t)).$$

- This is an ordinary differential equation!
- *Punchline:* To determine, at least locally, an integral curve for a vector field, one has to solve a differential equation.
- Solving differential equations is generally impossible. However, from the theory of differential equations we may assert that integral curves exist (for small times) and are unique.
- However, the new idea here is that of thinking of a solution to a differential equation as a curve:

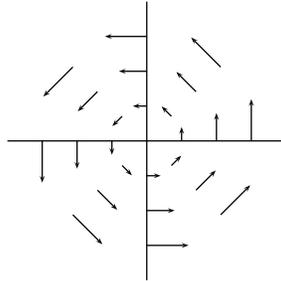


- This more geometric interpretation of the solution to a differential equation is a powerful one, as we shall see.

Example 5.3 Let us take $M = \mathbb{R}^2$ and take (x, y) to be the usual Cartesian coordinates. We define a vector field X by defining its coordinate representative in these coordinates:

$$(x, y) \mapsto ((x, y), (-y, x)).$$

Let us “draw” this vector field to get some idea of what it says:



From the picture, it seems reasonable that integral curves should be circles. Let us check this by writing down the differential equation, and seeing if we can solve it.

The differential equation is

$$\dot{x}(t) = -y(t), \quad \dot{y}(t) = x(t).$$

This is actually a linear differential equation, and we know how to solve it. The solution is

$$(x(t), y(t)) = (x(0) \cos t - y(0) \sin t, x(0) \sin t + y(0) \cos t).$$

Since $x(t)^2 + y(t)^2 = x(0)^2 + y(0)^2$, we can conclude that indeed the integral curves are circles. \square

- It is possible that an integral curve at x cannot be defined for all time. However, it is defined for small times, and by extending the interval on which an integral curve can be extended to be as large as possible, we obtain an interval which we denote by $I(X, x)$. The integral curve defined on $I(X, x)$ is called the *maximal integral curve* of X through x .
- The *domain* of X is the set

$$\text{dom}(X) = \{(t, x) \in \mathbb{R} \times M \mid t \in I(X, x)\}.$$

- For $(t, x) \in \text{dom}(X)$, denote by $\Phi_t^X(x)$ the point in M given by $\gamma(t)$, where γ is the maximal integral curve for X through x . The map $(t, x) \mapsto \Phi_t^X(x)$ is called the *flow* for X .

Example 5.4 (Vector field on \mathbb{R}^2 cont'd) We again take $M = \mathbb{R}^2$, and we let X be the vector field defined in Example 5.3. Note that all integral curves can be defined for all time. Thus $I(X, (x, y)) = \mathbb{R}$ for each $(x, y) \in M$. Therefore,

$\text{dom}(X) = \mathbb{R} \times M$. The flow is given by

$$(t, (x, y)) \mapsto (x \cos t - y \sin t, x \sin t + y \cos t).$$

The symbol $\Phi_t^X(x, y)$ is represented by

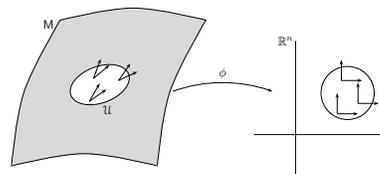
$$\Phi_t^X(x, y) = (x \cos t - y \sin t, x \sin t + y \cos t).$$

If one fixes t and thinks of the map $(x, y) \mapsto \Phi_t^X(x, y)$ as being a transformation of M , then this transformation is simply a rotation by angle t . This, again, is a more geometric way of thinking of a solution to a differential equation. \square

Vector fields (cont'd)

5.2. Some coordinate notation

- We next introduce some notation concerning vector fields that, while seemingly weird at first, is actually extremely useful.
- Let X be a vector field on M and let (\mathcal{U}, ϕ) be a chart with coordinates (x^1, \dots, x^n) .
- Define vector fields E_1, \dots, E_n by asking that the local representative of E_i in this chart be $x \mapsto (x, e_i)$, where e_i is the i th standard basis vector.
- Since the standard basis is...er... a basis, for each $x \in \mathcal{U}$, $\{E_1(x), \dots, E_n(x)\}$ is a basis for $T_x M$.
- *Notation:* $E_i = \frac{\partial}{\partial x^i}$, $i \in \{1, \dots, n\}$.
- The notation seems weird initially, but we shall see shortly why it is so useful.



- Note that if X is a vector field defined on \mathcal{U} , then we may write

$$X(x) = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i} = \underbrace{X^i \frac{\partial}{\partial x^i}}_{\text{what does this mean?}}. \quad (3)$$

- The last term in the previous equation introduces some new notation we shall make extensive use of. Namely, we have omitted the summation sign.
- The convention being employed here is called the *Einstein summation convention*, and it says that in an expression where repeated indices occur, summation over these indices is implied.
- Some rules for the summation convention:
 1. summation always occurs over one index that is a superscript, and one that is a subscript (a superscript (resp. subscript) in the denominator of an expression is taken to be a subscript (resp. superscript));
 2. in an equality where the summation convention is being used, the “free indices” (i.e., those not being summed) should agree;
 3. no index should appear more than twice in the same expression.

- We shall see how these rules come up as we go along.
- In (3), the functions X^1, \dots, X^n on \mathcal{U} are the same components of X we defined in the last lecture.

Change of basis formulae for vector fields

- Let X be a vector field, let (x^1, \dots, x^n) and $(\tilde{x}^1, \dots, \tilde{x}^n)$ be coordinates, and let X^1, \dots, X^n and $\tilde{X}^1, \dots, \tilde{X}^n$ be the components of X in the two sets of coordinates.
- Since $X(x) \in T_x M$, it follows immediately that the components of X transform in the same way as do the components of tangent vectors.
- We look back and see that this means that

$$\tilde{X}^i = \frac{\partial \tilde{x}^i}{\partial x^j} X^j, \quad i \in \{1, \dots, n\}.$$

Note that the summation convention is in force.

- Let us also investigate how the basis vector fields $\frac{\partial}{\partial \tilde{x}^i}$ are related to the basis vector fields $\frac{\partial}{\partial x^i}$.
- If the notation makes sense, then it should be the case that one can use the Chain Rule:

$$\frac{\partial}{\partial \tilde{x}^i} = \frac{\partial x^j}{\partial \tilde{x}^i} \frac{\partial}{\partial x^j}.$$

- To check that this *is* the actual transformation rule, we should check that

$$X = X^i \frac{\partial}{\partial x^i} = \tilde{X}^i \frac{\partial}{\partial \tilde{x}^i}.$$

Let us verify this:

$$\tilde{X}^i \frac{\partial}{\partial \tilde{x}^i} = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{\partial x^k}{\partial \tilde{x}^i} X^j \frac{\partial}{\partial x^k}.$$

Note that $\frac{\partial \tilde{x}^i}{\partial x^j}$ are the components of the Jacobian of the overlap map going from coordinates x to \tilde{x} , and $\frac{\partial x^k}{\partial \tilde{x}^i}$ are the components of the Jacobian of the overlap map going from coordinates x to \tilde{x} . These matrices are inverses of one another.

This means that

$$\frac{\partial \tilde{x}^i}{\partial x^j} \frac{\partial x^k}{\partial \tilde{x}^i} = \delta_k^j = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

Note that δ_k^j are then the components of the identity matrix. This symbol is called the **Kronecker delta**. Now we have

$$\tilde{X}^i \frac{\partial}{\partial \tilde{x}^i} = \delta_k^j X^j \frac{\partial}{\partial x^k} = X^k \frac{\partial}{\partial x^k},$$

using the definition of the Kronecker delta. Thus, after the dust settles, we do indeed see that the proposed change of coordinates rule for the basis vector fields makes sense.

Example 5.5 Let us look at an example to illustrate these ideas. We take $M = \mathbb{R}^2$, and the vector field X we dealt with in the preceding lecture. In the notation of last time, we said that X had local representative

$$(x, y) \mapsto ((x, y), (-y, x))$$

in Cartesian coordinates. In our present notation, we would write

$$X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.$$

We also wish to write this vector field in polar coordinates. We do this in two ways, first by transforming the components of X , then by transforming the basis vectors.

First the components transformation rule. Let us write (X_x, X_y) as the components

of X in Cartesian coordinates (i.e., $X_x = -y$ and $X_y = x$) and (X_r, X_θ) as the components of X in polar coordinates. We have

$$\begin{aligned} X_r &= \frac{\partial r}{\partial x} X_x + \frac{\partial r}{\partial y} X_y = \frac{x}{\sqrt{x^2 + y^2}}(-y) + \frac{y}{\sqrt{x^2 + y^2}}(x) = 0, \\ X_\theta &= \frac{\partial \theta}{\partial x} X_x + \frac{\partial \theta}{\partial y} X_y = \frac{-y}{x^2 + y^2}(-y) + \frac{x}{x^2 + y^2}(x) = 1. \end{aligned}$$

Therefore,

$$X = 0 \frac{\partial}{\partial r} + 1 \frac{\partial}{\partial \theta}.$$

Now let us change the basis vectors. We have

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta} \\ &= \frac{x}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial r} - \frac{y}{x^2 + y^2} \frac{\partial}{\partial \theta} \\ &= \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y} \frac{\partial}{\partial \theta} \\ &= \frac{y}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial r} + \frac{x}{x^2 + y^2} \frac{\partial}{\partial \theta} \\ &= \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}. \end{aligned}$$

Substituting this into the expression for X we get

$$X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} = 0 \frac{\partial}{\partial r} + 1 \frac{\partial}{\partial \theta},$$

as, of course, we must. □

- In some sense, the change of coordinates rules for vector fields *define* vector fields. That is to say, they provide us with a means of identifying whether an object is a vector field.
- As a specific instance of this, consider the following question: “For a function $f: M \rightarrow \mathbb{R}$, is the object represented in coordinates as having components $(\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^n})$ a vector field?”
- To answer this question, we need only check whether the object obeys the change of coordinates rule for vector fields.
- By the Chain Rule we have

$$\frac{\partial f}{\partial \tilde{x}^i} = \frac{\partial x^j}{\partial \tilde{x}^i} \frac{\partial f}{\partial x^j}.$$

If the object were to be a vector field, however, it would have to satisfy

$$\frac{\partial f}{\partial \tilde{x}^i} = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{\partial f}{\partial x^j}.$$

Note that this formula wrecks havoc with the summation convention, which itself is cause for suspicion.

- In any event, the object does not transform like a vector field, so cannot be a vector field. So what is it?...

6. Some linear algebra

- We now engage in a short diversion into linear algebra, taking a new look at things you have probably seen before.

6.1. Dual spaces

- Let V be a finite-dimensional \mathbb{R} -vector space.
- By V^* , denote the set of linear maps from V to \mathbb{R} . This is the *dual* of V .
- Recall that the set of linear maps from an n -dimensional vector space to an m -dimensional vector space forms a vector space of dimension nm (think matrices, if this helps). Thus V^* is an n -dimensional vector space if $\dim(V) = n$.
- We will exhibit a basis for V^* given a basis $\{e_1, \dots, e_n\}$ for V .
- For $i \in \{1, \dots, n\}$, define $e^i \in V^*$ by

$$e^i(v) = e^i(v^j e_j) = v^i.$$

Thus e^i picks out the i th component of $v \in V$ in the basis $\{e_1, \dots, e_n\}$.

- **Notation:** We number vectors in V with *subscripts*, and we number vectors in V^* with *superscripts*. This will be important in the use of the summation convention as we go along.

Proposition 6.1 *The set $\{e^1, \dots, e^n\}$ is a basis for V^* , called the **dual basis** to $\{e_1, \dots, e_n\}$.*

Proof: We will show linear independence of the dual basis, as well as show that it spans V^* . In actuality, we need only show one of these two properties. (Why?)

For linear independence, suppose that we have constants $c_1, \dots, c_n \in \mathbb{R}$ so that $c_i e^i = 0$. Note that $e^i(e_j) = \delta_j^i$ (the i th component of the j th basis vector is zero, unless $i = j$, in which case it is one). Therefore

$$0 = c^i e^i(e_j) = c_i \delta_j^i = c_j.$$

Since this holds for all $j \in \{1, \dots, n\}$, it follows that $\{e^1, \dots, e^n\}$ is linearly independent.

Now we show that the dual basis spans V^* . Let $\alpha \in V^*$ and define $\alpha_i = \alpha(e_i) \in \mathbb{R}$, $i \in \{1, \dots, n\}$. For $v \in V$ we have

$$\alpha(v) = \alpha(v^i e_i) = v^i \alpha(e_i) = \alpha_i e^i(v).$$

Since this must hold for all $v \in V$, it follows that $\alpha = \alpha_i e^i$. Thus $\{e^1, \dots, e^n\}$ spans V^* , as desired. ■

- If $\alpha \in V^*$, then we write $\alpha = \alpha_i e^i$, and $\alpha^1, \dots, \alpha^n$ are the **components** of α in the dual basis.
- **Notation:** Note that the components of a vector are indexed with *superscripts*, while the components of a dual vector are indexed with *subscripts*.
- Note that if $\alpha \in V^*$ and if $v \in V$ we have

$$\alpha(v) = \alpha_i e^i(v^j e_j) = \alpha_i v^j e^i(e_j) = \alpha_i v^j \delta_j^i = \underbrace{\alpha_i v^i}_{\text{not a dot product!}}. \quad (4)$$

- The last term looks alluringly like a dot product. It is not. A dot product takes **two vectors** and gives a number. The expression in (4) is that of an element of V^* acting on an element of V . This is a cause for confusion. Unconfuse yourself.

- To clarify things, let us look at an example.

Example 6.2 Take $V = \mathbb{R}^n$, and let $\{e_1, \dots, e_n\}$ be the standard basis. Let us follow the usual practice, and think of vectors in \mathbb{R}^n as column vectors. Then, a linear map from \mathbb{R}^n to \mathbb{R}^m is to be thought of as an $m \times n$ matrix. In particular, elements of $(\mathbb{R}^n)^*$ are to be thought of as $1 \times n$ matrices, i.e., as row vectors. Thus we write a typical element in \mathbb{R}^n as

$$\mathbf{v} = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \quad (5)$$

and we write a typical element in $(\mathbb{R}^n)^*$ as

$$\boldsymbol{\alpha} = [\alpha_1 \quad \cdots \quad \alpha_n]. \quad (6)$$

The basis for $(\mathbb{R}^n)^*$ dual to the standard basis is then readily seen to be

$$\mathbf{e}^1 = [1 \quad 0 \quad \cdots \quad 0], \quad \mathbf{e}^2 = [0 \quad 1 \quad \cdots \quad 0], \quad \dots, \quad \mathbf{e}^n = [0 \quad 0 \quad \cdots \quad 1].$$

With $\boldsymbol{\alpha}$ as in (6) and with \mathbf{v} as in (5), we have

$$\boldsymbol{\alpha}(\mathbf{v}) = \alpha_i v^i.$$

Again, note that this is not a dot product, since $\boldsymbol{\alpha}$ is not a column vector, i.e., not an element of \mathbb{R}^n . \square

- **Notation:** We shall use the following notation interchangeably for the same thing:

$$\boldsymbol{\alpha}(\mathbf{v}), \quad \langle \boldsymbol{\alpha}; \mathbf{v} \rangle, \quad \boldsymbol{\alpha} \cdot \mathbf{v}.$$

Again, the last bit of notation makes you think dot product, but don't do this. We will have alternative notation for what you know as the dot product.

6.2. Symmetric bilinear maps

- We now introduce a concept that is somewhat familiar to you, but we will think of it a little more carefully than perhaps you are used to doing.

Definition 6.3 Let V be a finite-dimensional \mathbb{R} -vector space.

(i) A **bilinear map** on V is a map $B: V \times V \rightarrow \mathbb{R}$ with the property that

$$B(c_1v_1 + c_2v_2, v_3) = c_1B(v_1, v_3) + c_2B(v_2, v_3),$$

for all $v_1, v_2, v_3 \in V$, and $c_1, c_2 \in \mathbb{R}$.

(ii) A bilinear map B is **symmetric** (resp. **skew-symmetric**) if

$$B(v_1, v_2) = B(v_2, v_1) \text{ (resp. } B(v_1, v_2) = -B(v_2, v_1)) \text{ for all } v_1, v_2 \in V.$$

The set of symmetric bilinear maps is denoted $\Sigma_2(V)$.

- Let $\{e_1, \dots, e_n\}$ be a basis for V . Given a bilinear map B , the n^2 numbers $B_{ij} = B(e_i, e_j)$, $i, j \in \{1, \dots, n\}$, are the **components** of B in the basis.
- Note that if $u = u^i e_i$ and $v = v^j e_j$, then

$$B(u, v) = B(u^i e_i, v^j e_j) = u^i v^j B(e_i, e_j) = B_{ij} u^i v^j.$$

Thus, to know the value of B on any two vectors, it suffices to know the components of B , along with the components of the vectors.

Some linear algebra (cont'd)

Example 6.4 Let $V = \mathbb{R}^n$ and denote by $\mathbb{G}_{\mathbb{R}^n} \in \Sigma_2(\mathbb{R}^n)$ the standard inner product on \mathbb{R}^n , i.e., the “dot product.” Let’s compute the components of $\mathbb{G}_{\mathbb{R}^n}$ with respect to the standard basis $\{e_1, \dots, e_n\}$:

$$\mathbb{G}_{\mathbb{R}^n}(e_i, e_j) = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Note that this gives

$$\mathbb{G}_{\mathbb{R}^n}(u, v) = \delta_{ij} u^i v^j,$$

which represents the “dot product” in our summation convention form. \square

6.3. Plain ol’ linear maps

- Let us revisit the venerable subject of linear maps, but now using our notation.
- Let U and V be \mathbb{R} -vector spaces with bases $\{f_1, \dots, f_m\}$ and $\{e_1, \dots, e_n\}$. Let $A \in L(U; V)$ (=the set of linear maps from U to V).

- The *components* of A in the bases are the nm numbers A_a^i , $a \in \{1, \dots, m\}$, $i \in \{1, \dots, n\}$, that satisfy

$$A(f_a) = A_a^i e_i.$$

- We would like to think of these as being elements in a matrix, so we must decide which index is the row index, and which is the column index.
- Here are two possible ways to think of this.
 1. A linear map from \mathbb{R}^m to \mathbb{R}^n is represented by a matrix with n rows and m columns. Thus the superscript should be the row index, and the subscript should be the column index.
 2. Let $u = u^a f_a$. Then $A(u) = A(u^a f_a) = A_a^i u^a e_i$. Thus the components of $A(u)$ are $A_a^i u^a$, $i \in \{1, \dots, n\}$. Summation in matrix/vector multiplication takes place over the columns of the matrix. Thus the subscript should be the column index, and the superscript the row index.

- In any case, we write

$$[A] = \begin{bmatrix} A_1^1 & A_2^1 & \cdots & A_m^1 \\ A_1^2 & A_2^2 & \cdots & A_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ A_1^n & A_2^n & \cdots & A_m^n \end{bmatrix},$$

which is the *matrix representative* for A .

- Note that the components of both a linear map from V to itself, and a symmetric bilinear map are represented by $n \times n$ matrices. However, *these are not the same sorts of objects!* This is evidenced, for example, by the location of the indices for the components of a linear map (one up and one down) and a symmetric bilinear map (both down).
- We also assemble the components of a bilinear map into a matrix by

$$[B] = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{bmatrix}.$$

6.4. Linear maps associated with bilinear maps

- Given a bilinear map $B: V \times V \rightarrow \mathbb{R}$, we define a map $B^b: V \rightarrow V^*$ (pronounced “B-flat”) by asking that, for $v \in V$, $B^b(v)$ satisfy $\langle B^b(v); u \rangle = B(u, v)$ for all $u \in V$.
- Make sure you understand how this defines a linear map $B^b \in L(V; V^*)$.
- Let $\{e_1, \dots, e_n\}$ be a basis for V with $\{e^1, \dots, e^n\}$ the dual basis. We wish to compute the components of B^b in these bases. By definition

$$B^b(e_i) = (B^b)_{ji} e^j$$

gives the components $(B^b)_{ji}$ of B^b . Let's see what these are. On the one hand,

$$\langle B^b(e_i); e_k \rangle = B(e_k, e_i) = B_{ki}.$$

On the other hand

$$\langle (B^b)_{ji} e^j; e_k \rangle = (B^b)_{ji} \langle e^j; e_k \rangle = (B^b)_{ji} \delta_k^j = (B^b)_{ki}.$$

Therefore, $(B^b)_{ji} = B_{ji}$. That is, the components of B^b are just those of B . This can be confusing.

- From this it follows that $B^b(v) = B_{ij} v^j e^i$. Thus the components of $B^b(v)$ are obtained by multiplying the vector of components of v by the matrix of components of B in the usual fashion.
- If B^b is invertible, then its inverse is denoted by $B^\sharp \in L(V^*; V)$ (pronounced “B-sharp”).
- By repeating the sort of computations as above, one can show that the components of B^\sharp form a matrix which is the inverse of the matrix of components of B^b , and that the components of $B^\sharp(\alpha)$ are obtained by multiplying the vector of components of α by the matrix of components of B^\sharp .
- Note that if \mathbb{G} is an inner product, then \mathbb{G}^b is invertible. Indeed, the matrix $[\mathbb{G}]$ is symmetric and positive-definite (this follows from the properties of an inner product). This means that all of its eigenvalues are real and positive, and therefore the matrix is invertible, giving invertibility of \mathbb{G}^b .

7. Kinetic energy and Riemannian metrics

- We now wish to go about defining the kinetic energy of an interconnected mechanical system.
- This will involve our determining the kinetic energy first for each of the component rigid bodies and particles, then summing these together. We first need to spend some time talking about rigid bodies.

7.1. Rigid bodies

- A *rigid body* is a pair (\mathcal{B}, μ) where $\mathcal{B} \subset \mathbb{R}^3$ is compact, and where μ is a “positive Borel measure on \mathbb{R}^3 whose support is equal to \mathcal{B} ,” and is called the *mass distribution*.
- Most often, we shall take $\mu = \rho dV$, where ρ is the “mass density” of the body, and dV is the volume element in \mathbb{R}^3 .
- However, if one wants to consider particles as rigid bodies (and it is convenient to do so), then we cannot have so simple an idea of what μ is. In this case, we think of μ as being a map on the (Borel) subsets of \mathbb{R}^3 into $[0, \infty[$, with $\mu(S)$ being the “mass” of the set S .

Examples 7.1 1. A particle is the rigid body $\mathcal{B} = \{\chi_0\}$ (this is the location of the particle) and a mass distribution defined by

$$\mu(S) = \begin{cases} m, & \chi_0 \in S, \\ 0, & \chi_0 \notin S, \end{cases}$$

where m is the mass of the particle.

2. Suppose we have two masses at points χ_1 and $\chi_2 \in \mathbb{R}^3$, and with masses m_1 and m_2 . The particles are constrained to have the same position relative to one another, and so form a rigid body. The mass distribution here would be

$$\mu(S) = \begin{cases} m_1 + m_2, & \chi_1, \chi_2 \in S, \\ m_1, & \chi_1 \in S, \chi_2 \notin S, \\ m_2, & \chi_1 \notin S, \chi_2 \in S, \\ 0, & \chi_1, \chi_2 \notin S. \end{cases}$$

3. A general rigid body will not be contained in any line, and there is not much one can say in general about its mass distribution. \square

7.2. Mass and center of mass

- The *mass* of (\mathcal{B}, μ) is

$$\mu(\mathcal{B}) \triangleq \int_{\mathbb{R}^3} d\mu.$$

- The *center of mass* of (\mathcal{B}, μ) is

$$\chi_c \triangleq \frac{1}{\mu(\mathcal{B})} \int_{\mathbb{R}^3} \chi d\mu.$$

Examples 7.2 1. The particle at χ_0 has center of mass

$$\chi_c = \frac{1}{m} \int_{\mathbb{R}^3} \chi d\mu = \frac{1}{m} m\chi_0 = \chi_0,$$

as expected.

2. The two particles have center of mass

$$\chi_c = \frac{1}{m_1 + m_2} \int_{\mathbb{R}^3} \chi d\mu = \frac{1}{m_1 + m_2} (m_1\chi_1 + m_2\chi_2),$$

again, as expected.

7.3. The inertia tensor

- Let $\chi_0 \in \mathbb{R}^3$. The *inertia tensor* for (\mathcal{B}, μ) about χ_0 is the linear map $\mathbb{I}_{\chi_0} \in L(\mathbb{R}^3; \mathbb{R}^3)$ given by

$$\mathbb{I}_{\chi_0}(\mathbf{u}) = \int_{\mathbb{R}^3} (\chi - \chi_0) \times (\mathbf{u} \times (\chi - \chi_0)) d\mu.$$

- We shall make two simplifying assumptions:
 1. $\chi_0 = \chi_c$: we take the moment of inertia about the center of mass;
 2. $\chi_c = \mathbf{0}$: we choose the body frame so that the origin is at the center of mass.

In this case we denote $\mathbb{I}_c = \mathbb{I}_{\chi_c}$, and have

$$\mathbb{I}_c(\mathbf{u}) = \int_{\mathbb{R}^3} \chi \times (\mathbf{u} \times \chi) d\mu.$$

- Let us record some nice properties of the inertia tensor.

Proposition 7.3 *We have*

- \mathbb{I}_c is a symmetric linear map with respect to the standard inner product on \mathbb{R}^3 , and
- $\mathbb{G}_{\mathbb{R}^3}(\mathbb{I}_c(\mathbf{u}, \mathbf{u})) \geq 0$.

Proof: We recall the vector identity

$$\mathbb{G}_{\mathbb{R}^3}(\mathbf{u}, \mathbf{v} \times \mathbf{w}) = \mathbb{G}_{\mathbb{R}^3}(\mathbf{w}, \mathbf{u} \times \mathbf{v}). \quad (7)$$

Using this identity twice we have

$$\begin{aligned} \mathbb{G}_{\mathbb{R}^3}(\mathbb{I}_c(\mathbf{u}), \mathbf{v}) &= \int_{\mathbb{R}^3} \mathbb{G}_{\mathbb{R}^3}(\boldsymbol{\chi} \times (\mathbf{u} \times \boldsymbol{\chi}), \mathbf{v}) \, d\mu \\ &= \int_{\mathbb{R}^3} \mathbb{G}_{\mathbb{R}^3}(\mathbf{u} \times \boldsymbol{\chi}, \mathbf{v} \times \boldsymbol{\chi}) \, d\mu \\ &= \int_{\mathcal{B}} \mathbb{G}_{\mathbb{R}^3}(\mathbf{u}, \boldsymbol{\chi} \times (\mathbf{v} \times \boldsymbol{\chi})) \, d\mu \\ &= \mathbb{G}_{\mathbb{R}^3}(\mathbf{u}, \mathbb{I}_x(\mathbf{v})), \end{aligned} \quad (8)$$

which gives the symmetry of \mathbb{I}_c . That \mathbb{I}_c is positive-semidefinite follows directly from (8). ■

- Note that it follows that
 1. all eigenvalues of \mathbb{I}_c are real and nonnegative, and
 2. the eigenvectors of \mathbb{I}_c form an orthogonal basis for \mathbb{R}^3 .

- The eigenvalues we denote by J_1, J_2, J_3 , and call these the *principal inertias*. The eigenvectors we denote by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, and call these the *principal axes*.
- The inertial ellipsoid, along with the mass, gives us all the essential inertial information about a rigid body.
- Here's an interesting experiment to perform at home.
 1. Take a rigid body with three distinct principal inertias. A good example of this is a book with three quite different length/width/thickness dimensions.
 2. Spin the body about the axis corresponding to the smallest principal inertia. Note that it spins about this axis.
 3. Spin the body about the axis corresponding to the largest principal inertia. Note that it spins about this axis.
 4. Spin the body about the axis corresponding to the intermediate principal inertia. Note that now the body does not just spin about this axis. The rotation about this axis is unstable.
- To understand the general motion of a body, look up *Poinsot's Theorem* on the web.

Kinetic energy and Riemannian metrics (cont'd)

- We shall not dwell much on the computation of the inertia tensor. Commonly shaped bodies have formulae for their moments of inertia that can be looked up. For really complicated bodies, measurement is the best approach.
- However, for the situations we consider in this course, a few comments are helpful.
 1. For a planar body, the vector orthogonal to the plane can be assumed to be a principal axis.
 2. If a body has an axis of symmetry (meaning its inertial ellipsoid is invariant under rotations about some axis), then
 - (a) this axis of symmetry is a principal axis, and
 - (b) the other two principal inertias are equal.

7.4. Kinetic energy of a rigid body

- The kinetic energy of a particle with mass m undergoing a motion described by $t \mapsto \mathbf{r}(t)$ is $\text{KE}(t) = \frac{1}{2}m\|\dot{\mathbf{r}}(t)\|_{\mathbb{R}^3}$.
- We wish to generalise this to a rigid body.
- We suppose the body is undergoing a motion described by $t \mapsto (\mathbf{R}(t), \mathbf{r}(t))$. A point in the body that is located at $\boldsymbol{\chi} \in \mathbb{R}^3$ relative to the body frame will be located at $\mathbf{x}(t) = \mathbf{r}(t) + \mathbf{R}\boldsymbol{\chi}(t)$ at time t .
- By analogy with the particle case, the *kinetic energy* is

$$\text{KE}(t) = \frac{1}{2} \int_{\mathcal{B}} \|\dot{\mathbf{r}}(t) + \dot{\mathbf{R}}(t)\boldsymbol{\chi}\|_{\mathbb{R}^3}^2 d\mu,$$

Kinetic energy and Riemannian metrics (cont'd)

- We wish to obtain a convenient expression for the kinetic energy.

Proposition 7.4 *Let a rigid body (\mathcal{B}, μ) undergo a motion specified by a differentiable curve $t \mapsto (\mathbf{R}(t), \mathbf{r}(t))$ as above. If*

$$\text{KE}_{\text{tran}}(t) = \frac{1}{2}\mu(\mathcal{B})\|\dot{\mathbf{r}}(t)\|_{\mathbb{R}^3}^2, \quad \text{KE}_{\text{rot}}(t) = \frac{1}{2}\mathbb{G}_{\mathbb{R}^3}(\mathbb{I}_c(\boldsymbol{\Omega}(t)), \boldsymbol{\Omega}(t)),$$

then $\text{KE}(t) = \text{KE}_{\text{tran}}(t) + \text{KE}_{\text{rot}}(t)$.

Proof: We compute

$$\begin{aligned} \text{KE}(t) &= \frac{1}{2} \int_{\mathcal{B}} \|\dot{\mathbf{r}}(t)\|_{\mathbb{R}^3}^2 d\mu + \frac{1}{2} \int_{\mathcal{B}} \|\dot{\mathbf{R}}(t)\boldsymbol{\chi}\|_{\mathbb{R}^3}^2 d\mu + \int_{\mathcal{B}} \mathbb{G}_{\mathbb{R}^3}(\dot{\mathbf{r}}(t), \dot{\mathbf{R}}(t)\boldsymbol{\chi}) d\mu \\ &= \frac{1}{2}\mu(\mathcal{B})\|\dot{\mathbf{r}}(t)\|_{\mathbb{R}^3}^2 + \frac{1}{2} \int_{\mathcal{B}} \|\dot{\mathbf{R}}(t)\boldsymbol{\chi}\|_{\mathbb{R}^3}^2 d\mu, \end{aligned}$$

using in the second line the fact that

$$\int_{\mathcal{B}} \boldsymbol{\chi} d\mu = \mathbf{0},$$

since the center of mass of the body is at the origin of the body frame. Thus the

result will follow if we can show that

$$\int_{\mathcal{B}} \|\dot{\mathbf{R}}(t)\boldsymbol{\chi}\|_{\mathbb{R}^3}^2 d\mu = \mathbb{G}_{\mathbb{R}^3}(\mathbb{I}_c(\boldsymbol{\Omega}), \boldsymbol{\Omega}(t)).$$

To this end we compute

$$\begin{aligned} \int_{\mathcal{B}} \|\dot{\mathbf{R}}(t)\boldsymbol{\chi}\|_{\mathbb{R}^3}^2 d\mu &= \int_{\mathcal{B}} \|\mathbf{R}(t)(\boldsymbol{\Omega} \times \boldsymbol{\chi})\|_{\mathbb{R}^3}^2 d\mu = \int_{\mathcal{B}} \|\boldsymbol{\Omega} \times \boldsymbol{\chi}\|_{\mathbb{R}^3}^2 d\mu \\ &= \int_{\mathcal{B}} \mathbb{G}_{\mathbb{R}^3}(\boldsymbol{\chi} \times (\boldsymbol{\Omega}(t) \times \boldsymbol{\chi}), \boldsymbol{\Omega}(t)) d\mu \\ &= \mathbb{G}_{\mathbb{R}^3}(\mathbb{I}_c(\boldsymbol{\Omega}), \boldsymbol{\Omega}(t)), \end{aligned}$$

where we have used the vector identity (7) and the fact that $\boldsymbol{\chi}_c = \mathbf{0}$. ■

- **Important note:** If the body frame is not located at the center of mass, the preceding result does not hold. That is to say, the kinetic energy does not generally decouple into a translational and rotational part.
- **Side fact:** If the body is fixed in space at some point (i.e., that point does not move), then one might place the body frame at this point. If one does this, the kinetic energy will be purely rotational, and computed using the inertia tensor about the fixed point.

7.5. Kinetic energy for an interconnected mechanical system

- Suppose we have an interconnected mechanical system comprised of rigid bodies $(\mathcal{B}_1, \mu_1), \dots, (\mathcal{B}_k, \mu_k)$, and with configuration manifold $Q \subset Q_{\text{free}}$.
- Now suppose that the system is in motion, meaning we have a curve $t \mapsto \gamma(t)$ in Q . Each rigid body will then be in motion, and so its kinetic energy will be computed as above. The total kinetic energy will be the sum of the kinetic energies for each body. We wish to make this a little more precise.
- Fix $a \in \{1, \dots, k\}$ and let $\Pi_a: Q \rightarrow \text{SO}(3) \times \mathbb{R}^3$ be the map that assigns the position of the a th body to the point $q \in Q$. Define $(\mathbf{R}_a(t), \mathbf{r}_a(t)) = \Pi_a \circ \gamma(t)$. Then the kinetic energy of the a th body is as computed above:

$$\text{KE}_a(t) = \frac{1}{2} \mu_a(\mathcal{B}_a) \|\dot{\mathbf{r}}_a(t)\|_{\mathbb{R}^3}^2 + \frac{1}{2} \mathbb{G}_{\mathbb{R}^3 \mathbb{I}_{c,a}}(\boldsymbol{\Omega}_a(t)) \boldsymbol{\Omega}_a(t).$$

- The total kinetic energy along γ is

$$\text{KE}_\gamma(t) = \sum_{a=1}^k \text{KE}_a(t).$$

- Note that the kinetic energy at time t for the system depends on the position at time t and the velocity at time t . Thus, it makes sense to speculate on the existence of a function $\text{KE}: \text{T}Q \rightarrow \mathbb{R}$ with the property that $\text{KE}(\gamma'(t)) = \text{KE}_\gamma(t)$.
- Indeed, it is easy to define such a function. Let $v_q \in \text{T}Q$ and let γ be a curve at q for which $v_q = [\dot{\gamma}]$. Then we simply define $\text{KE}(v_q) = \text{KE}_\gamma(0)$. This is the **kinetic energy** for the interconnected mechanical system.
- **Question:** For fixed $q \in Q$, what properties does the function KE have when restricted to $\text{T}_q Q$?
- **Answers:**
 1. It is quadratic in velocity, since the rigid body terms are quadratic in velocity.
 2. It is nonnegative, again since the rigid body terms are nonnegative.

Therefore, $\text{KE}(v_q) = \frac{1}{2} \mathbb{G}(q)(v_q, v_q)$, where $\mathbb{G}(q)$ is a symmetric bilinear map on $\text{T}_q Q$ having the property that $\mathbb{G}(q)(v_q, v_q) \geq 0$ for all v_q .
- If the modeling of the system has been done in a “reasonable” way, it turns out that $\mathbb{G}(q)(v_q, v_q) > 0$ whenever $v_q \neq 0$. “Reasonable” essentially means that there are no degrees of freedom of the system that have no inertia associated to them (e.g., particles with rotational degrees of freedom).

- Motivated by the preceding description of the kinetic energy for an interconnected mechanical system, we have the following important concept.

Definition 7.5 On a manifold Q (not necessarily the configuration manifold for a free mechanical system) a *Riemannian metric* is a smooth assignment of an inner product $\mathbb{G}(q)$ to each point $q \in Q$. □

- *Punchline:* Every interconnected mechanical system possesses a natural Riemannian metric \mathbb{G} with the property that $\text{KE}(v_q) = \frac{1}{2}\mathbb{G}(v_q, v_q)$. This Riemannian metric is called the *kinetic energy metric*.
- The kinetic energy metric carries the information about the inertial properties of the system.
- For the remainder of the course, the kinetic energy metric, and entities associated with it, will be extremely important.

Kinetic energy and Riemannian metrics (cont'd)

7.6. Coordinate representations for covectors and Riemannian metrics

- Let Q be a manifold. The dual space to the tangent space $T_q Q$ is denoted $T_q^* Q$, and is called the *cotangent space*. Elements of $T_q^* Q$ are called *cotangent vectors*. The collection $T^* Q = \cup_{q \in Q} T_q^* Q$ is called the *cotangent bundle*.
- We have seen that a vector field is an assignment of a tangent vector to each point in Q . A *covector field* is an assignment
- Now suppose that we have coordinates (q^1, \dots, q^n) . We had the notation $\frac{\partial}{\partial q^i}$, $i \in \{1, \dots, n\}$, for a basis of vector fields in the given set of coordinates.
- We have seen the idea of a dual basis corresponding to a basis for a vector space. We do the same for tangent spaces, using funny notation for this.
- The basis dual to $\{\frac{\partial}{\partial q^1}(q), \dots, \frac{\partial}{\partial q^n}(q)\}$ we denote by $\{dq^1(q), \dots, dq^n(q)\}$. The reason for this notation will become clear later.
- A covector field α is then written in coordinates as $\alpha = \alpha_i dq^i$, where $\alpha_1, \dots, \alpha_n$ are the *components* of α .
- This is all just like we did when we talked about linear algebra.

- Now suppose that \mathbb{G} is a Riemannian metric on Q .
- The *components* of \mathbb{G} in coordinates (q^1, \dots, q^n) are the n^2 numbers $\mathbb{G}_{ij} = \mathbb{G}\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\right)$.
- The way we write \mathbb{G} in coordinates is

$$\mathbb{G} = \mathbb{G}_{ij} dq^i \otimes dq^j. \quad (9)$$

- The symbol \otimes is called the *tensor product*.
- In this course, the only use we will make of the tensor product is in writing the expression (9).
- However, here's a useful fact to help you think about what the $dq^i \otimes dq^j$ are:
The set $\{dq^i(q) \otimes dq^j(q) \mid i, j \in \{1, \dots, n\}\}$ is a basis for the set of bilinear maps from $T_q Q \times T_q Q$ to \mathbb{R} .

- Now we look at how this works in some examples.

Example 7.6 (Planar rigid body cont'd) We had previously determined that the configuration manifold for the system is $Q = \mathbb{S}^1 \times \mathbb{R}^2$. In Example 2.13 we made some choices concerning spatial and body frames, so these should be recalled. In Example 2.16 we introduced coordinates (θ, x, y) for the system, so these should be recalled.

With the preceding recollections, the coordinate form for the map $\Pi_1 : Q \rightarrow \text{SO}(3) \times \mathbb{R}^3$ that defines the position of the body at a given configuration is by

$$(\theta, x, y) \mapsto \left(\underbrace{\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_R, \underbrace{(x, y, 0)}_r \right).$$

Now one considers a curve $t \mapsto (\theta(t), x(t), y(t))$ in these coordinates and computes

$$\dot{\mathbf{R}}(t) = \begin{bmatrix} -\sin \theta(t) & -\cos \theta(t) & 0 \\ \cos \theta(t) & -\sin \theta(t) & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta}, \quad \dot{\mathbf{r}}(t) = (\dot{x}(t), \dot{y}(t), 0).$$

It is then an elementary computation to get

$$\widehat{\boldsymbol{\Omega}} = \mathbf{R}^T(t) \dot{\mathbf{R}}(t) = \dot{\theta} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

giving $\boldsymbol{\Omega}(t) = (0, 0, \dot{\theta})$.

Now we need the inertia tensor for the body. Since it is planar, and since \mathbf{b}_3 is orthogonal to the plane of motion, we can suppose that \mathbf{b}_3 is a principal axis, and denote the corresponding principal inertia by J . The exact character of the other

principal axes and principal inertias is irrelevant, so we can simply assume that

$$\mathbb{I}_c = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & J \end{bmatrix}.$$

(Why can we assume that the (1, 3), (2, 3), (3, 1), and (3, 2) entries are zero?) The mass of the body is denoted m . Now one simply uses Proposition 7.4 to compute

$$\text{KE} = \frac{1}{2} m \|\dot{\mathbf{r}}(t)\|_{\mathbb{R}^3}^2 + \frac{1}{2} \mathbb{G}_{\mathbb{R}^3}(\mathbb{I}_c(\boldsymbol{\Omega}(t)), \boldsymbol{\Omega}(t)) = \underbrace{\frac{1}{2} m (\dot{x}^2 + \dot{y}^2)}_{\text{KE}_{\text{tran}}} + \underbrace{\frac{1}{2} J \dot{\theta}^2}_{\text{KE}_{\text{rot}}}.$$

One can now write this as a function on TQ in coordinates as

$$\text{KE} = \frac{1}{2} m (v_x^2 + v_y^2) + \frac{1}{2} J v_\theta^2.$$

To compute the kinetic energy metric in coordinates we first compute the matrix with components

$$\mathbb{G}_{ij} = \frac{\partial^2 \text{KE}}{\partial v^i \partial v^j}, \quad i, j \in \{1, 2, 3\}.$$

This matrix we write as

$$[\mathbb{G}] = \begin{bmatrix} J & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix},$$

which is the *matrix representative* for the kinetic energy metric. To write the kinetic energy metric as a geometric object in coordinates, we use the notation (9):

$$\mathbb{G} = Jd\theta \otimes d\theta + m dx \otimes dx + m dy \otimes dy. \quad \square$$

Example 7.7 (Two-link manipulator (cont'd)) We refer to Examples 2.14 and 2.18 for a review of the description of $Q = \mathbb{S}^1 \times \mathbb{S}^1$ and the coordinates (θ_1, θ_2) .

The maps $\Pi_1, \Pi_2: Q \rightarrow SO(3) \times \mathbb{R}^3$ defining the positions of the body are readily seen to have the coordinate forms $(\theta_1, \theta_2) \mapsto (\mathbf{R}_a, \mathbf{r}_a)$, $a \in \{1, 2\}$, where

$$\mathbf{R}_a = \begin{bmatrix} \cos \theta_a & -\sin \theta_a & 0 \\ \sin \theta_a & \cos \theta_a & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad a \in \{1, 2\},$$

$$\mathbf{r}_1 = (\tfrac{1}{2}\ell_1 \cos \theta_1, \tfrac{1}{2}\ell_1 \sin \theta_1, 0),$$

$$\mathbf{r}_2 = (\ell_1 \cos \theta_1 + \tfrac{1}{2}\ell_2 \cos \theta_2, \ell_1 \sin \theta_1 + \tfrac{1}{2}\ell_2 \sin \theta_2, 0).$$

One then readily computes the body angular velocities as $\boldsymbol{\Omega}_a = (0, 0, \dot{\theta}_1)$, $a \in \{1, 2\}$.

We next need the inertia tensors. The motion is planar, and \mathbf{b}_3 is orthogonal to the plane of motion, so \mathbf{b}_3 is a principal axis for the inertia tensor of each body. Let J_1 and J_2 denote the principal inertias. As with the planar body example above, the remaining principal axes and principal inertias are not relevant. Now one simply grinds out the kinetic energy metric, using the formulae

$$\text{KE} = \frac{1}{2}m_1 \|\dot{\mathbf{r}}_1\|_{\mathbb{R}^3}^2 + \frac{1}{2}m_2 \|\dot{\mathbf{r}}_2\|_{\mathbb{R}^3}^2 + \frac{1}{2}\mathbb{G}_{\mathbb{R}^3}(\mathbb{I}_{c,1}(\boldsymbol{\Omega}_1), \boldsymbol{\Omega}_1) + \frac{1}{2}\mathbb{G}_{\mathbb{R}^3}(\mathbb{I}_{c,2}(\boldsymbol{\Omega}_2), \boldsymbol{\Omega}_2)$$

and

$$\mathbb{G}_{ij} = \frac{\partial^2 \text{KE}}{\partial v^i \partial v^j}.$$

After the dust settles one has

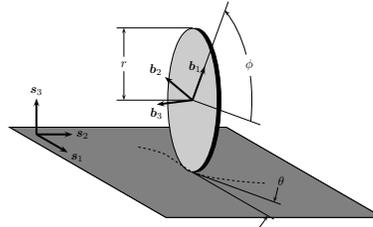
$$(J_1 + \frac{1}{4}(m_1 + 4m_2)\ell_1^2)d\theta_1 \otimes d\theta_1 + \frac{1}{2}m_2\ell_1\ell_2 \cos(\theta_1 - \theta_2)d\theta_1 \otimes d\theta_2$$

$$+ \frac{1}{2}m_2\ell_1\ell_2 \cos(\theta_1 - \theta_2)d\theta_2 \otimes d\theta_1 + (J_2 + \frac{1}{4}m_2\ell_2^2)d\theta_2 \otimes d\theta_2. \quad \square$$

Kinetic energy and Riemannian metrics (cont'd)

Example 7.8 (Rolling disk (cont'd))

We have $Q = \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$,
 and coordinates (x, y, θ, ϕ) as indicated
 in the figure. As usual, the key ingredient is the
 map $\Pi_1: Q \rightarrow \text{SO}(3) \times \mathbb{R}^3$. Here we reap the
 rewards of our careful attention to defining the
 configuration manifold, and we see that in coordinates Π_1 is given by



$$(x, y, \theta, \phi) \mapsto \left(\underbrace{\begin{bmatrix} \cos \phi \cos \theta & -\sin \phi \cos \theta & \sin \theta \\ \cos \phi \sin \theta & -\sin \phi \sin \theta & -\cos \theta \\ \sin \phi & \cos \phi & 0 \end{bmatrix}}_{\mathbf{R}}, \underbrace{(x, y, r)}_{\mathbf{r}} \right).$$

This then gives, by a direct computation, $\boldsymbol{\Omega} = (\sin \phi \dot{\theta}, -\cos \phi \dot{\theta}, \dot{\phi})$.

We need the inertia tensor. It is reasonable to assume that \mathbf{b}_3 is an axis of symmetry for the body. Therefore, it is a principal axis, and the remaining two principal inertias are equal. Thus we have

$$\mathbb{I}_c = \begin{bmatrix} J_{\text{spin}} & 0 & 0 \\ 0 & J_{\text{spin}} & 0 \\ 0 & 0 & J_{\text{roll}} \end{bmatrix},$$

for $J_{\text{spin}}, J_{\text{roll}} > 0$. Now we simply calculate

$$\text{KE} = \frac{1}{2} m \|\dot{\mathbf{r}}\|_{\mathbb{R}^3}^2 + \frac{1}{2} \mathbb{G}_{\mathbb{R}^3}(\mathbb{I}_c(\boldsymbol{\Omega}), \boldsymbol{\Omega})$$

and

$$\mathbb{G}_{ij} = \frac{\partial^2 \text{KE}}{\partial v^i \partial v^j}.$$

The result is

$$\mathbb{G} = m dx \otimes dx + m dy \otimes dy + J_{\text{spin}} d\theta \otimes d\theta + J_{\text{roll}} d\phi \otimes d\phi.$$

Note that the spin and roll kinetic energies decouple. Would you have guess this? \square

7.7. Changes of coordinate for covector fields and Riemannian metrics

- We have seen how basis vector fields and components of vector fields change when coordinates change. Let's do the same for covector fields and Riemannian metrics
- To deduce the change of coordinates formulae for covector fields, we use the fact that they act on vector fields, and we know the change of coordinate formulae for vector fields.
- Thus let (q^1, \dots, q^n) and $(\tilde{q}^1, \dots, \tilde{q}^n)$ be coordinates for Q , and let α and X be a covector field and a vector field, respectively. We shall write

$$X = X^i \frac{\partial}{\partial q^i} = \tilde{X}^i \frac{\partial}{\partial \tilde{q}^i} \quad \alpha = \alpha_i dq^i = \tilde{\alpha}_i d\tilde{q}^i.$$

- We know that

$$\tilde{X}^i = \frac{\partial \tilde{q}^i}{\partial q^j} X^j, \quad \frac{\partial}{\partial \tilde{q}^i} = \frac{\partial q^j}{\partial \tilde{q}^i} \frac{\partial}{\partial q^j}.$$

- We also know that $\alpha(X) = \alpha_i X^i = \tilde{\alpha}_i \tilde{X}^i$.

- We therefore compute

$$\alpha_i X^i = \alpha_i \frac{\partial \tilde{q}^i}{\partial q^j} \tilde{X}^j = \tilde{\alpha}_j \tilde{X}^j,$$

from which we deduce that $\tilde{\alpha}_j = \frac{\partial q^i}{\partial \tilde{q}^j} \alpha_i$.

- We also know that

$$\alpha = \alpha_i dq^i = \tilde{\alpha}_j \frac{\partial \tilde{q}^j}{\partial q^i} dq^i = \tilde{\alpha}_j d\tilde{q}^j,$$

from which we deduce that $d\tilde{q}^j = \frac{\partial \tilde{q}^j}{\partial q^i} dq^i$. Note that this is just how the coordinate change should work, if you think of the dq 's as being "infinitesimals."

- Recall that the mnemonic for the change of coordinate formula for vector fields is that the components should change like velocity changes using the Chain Rule.
- Is there a similar rule for the components of a covector field?
- To answer this question, recall that we had considered the object which in coordinates had components $(\frac{\partial f}{\partial q^1}, \dots, \frac{\partial f}{\partial q^n})$.

- We had decided that these components did not obey the transformation rule for vector fields, and so they could not be the components of a vector field. However, we *do* have

$$\frac{\partial f}{\partial \tilde{q}^i} = \frac{\partial q^j}{\partial \tilde{q}^i} \frac{\partial f}{\partial q^j},$$

simply by the Chain Rule.

- This then gives us two things.
 1. The object with components $(\frac{\partial f}{\partial q^1}, \dots, \frac{\partial f}{\partial q^n})$ is a covector field, since its components obey the right transformation rule for covector fields. This covector field we call the *differential* of f .
 2. The mnemonic for the transformation rule for the components of a covector is that they change like the partial derivatives of a function using the Chain Rule.
- Now we consider how the bases and components of a Riemannian metric act with respect to changes of coordinates.
- We consider coordinates (q^1, \dots, q^n) and $(\tilde{q}^1, \dots, \tilde{q}^n)$ and a Riemannian \mathbb{G} for which we can thus write

$$\mathbb{G} = \mathbb{G}_{ij} dq^i \otimes dq^j = \tilde{\mathbb{G}}_{k\ell} d\tilde{q}^k \otimes d\tilde{q}^\ell.$$

- We wish to compute the change of coordinate formulae for the components $\tilde{\mathbb{G}}_{k\ell}$ for got the basis vectors (for the set of bilinear maps) $d\tilde{q}^k \otimes d\tilde{q}^\ell$.
- The change of coordinate rule for the basis vectors is inherited from that for covector fields. Precisely,

$$d\tilde{q}^k \otimes d\tilde{q}^\ell = \left(\frac{\partial \tilde{q}^k}{\partial q^i} dq^i \right) \otimes \left(\frac{\partial \tilde{q}^\ell}{\partial q^j} dq^j \right) = \frac{\partial \tilde{q}^k}{\partial q^i} \frac{\partial \tilde{q}^\ell}{\partial q^j} dq^i \otimes dq^j.$$

- Note that we have implicitly used the fact that the tensor product \otimes is linear.
- The rule for change of components is now inherited from the change of bases. We have

$$\mathbb{G} = \mathbb{G}_{ij} dq^i \otimes dq^j = \mathbb{G}_{ij} \frac{\partial q^i}{\partial \tilde{q}^k} \frac{\partial q^j}{\partial \tilde{q}^\ell} d\tilde{q}^k \otimes d\tilde{q}^\ell,$$

from which we deduce that

$$\tilde{\mathbb{G}}_{k\ell} = \frac{\partial q^i}{\partial \tilde{q}^k} \frac{\partial q^j}{\partial \tilde{q}^\ell} \mathbb{G}_{ij}.$$

Kinetic energy and Riemannian metrics (cont'd)

- Let us look at some examples that illustrate the change of coordinate formulae for covector fields and Riemannian metrics.

Example 7.9 Take $Q = \mathbb{R}^2$ and let (x, y) and (r, θ) be Cartesian and polar coordinates, respectively. Define $f: Q \rightarrow \mathbb{R}$ by $f(x, y) = x^2 + y^2$. We wish to see how the covector field df , the differential of f , looks in both sets of coordinates.

There are two ways to do this. First, one can write the function in both sets of coordinates, and then take the differential. This is very easy. In Cartesian coordinates the local representative of f is $(x, y) \mapsto x^2 + y^2$, so that $df = 2x dx + 2y dy$. In polar coordinates the local representative of the function is $(r, \theta) \mapsto r^2$, so that $df = 2r dr + 0 d\theta$.

We can also compute, using the change of coordinate formulae, the components or the basis covector fields, in one coordinate system given those in the other. Let us illustrate how this goes, first using the change of coordinate rule for components of covector fields. We write

$$df = \alpha_x dx + \alpha_y dy = \alpha_r dr + \alpha_\theta d\theta,$$

where we wish to compute α_r and α_θ in terms of $\alpha_x = 2x$ and $\alpha_y = 2y$. We have

$$\begin{aligned} \alpha_r &= \frac{\partial x}{\partial r} \alpha_x + \frac{\partial y}{\partial r} \alpha_y = (\cos \theta)(2r \cos \theta) + (\sin \theta)(2r \sin \theta) = 2r \\ \alpha_\theta &= \frac{\partial x}{\partial \theta} \alpha_x + \frac{\partial y}{\partial \theta} \alpha_y = (-r \sin \theta)(2r \cos \theta) + (r \cos \theta)(2r \sin \theta) = 0. \end{aligned}$$

Thus $df = 2r dr + 0 d\theta$, as desired.

We can also use the change of coordinate rule for the basis. This calculation goes as follows:

$$\begin{aligned} df &= \alpha_x dx + \alpha_y dy \\ &= (2r \cos \theta) \left(\frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta \right) + (2r \sin \theta) \left(\frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta \right) \\ &= (2r \cos \theta) (\cos \theta dr - r \sin \theta d\theta) + (2r \sin \theta) (\sin \theta dr + r \cos \theta d\theta) \\ &= 2r dr + 0 d\theta, \end{aligned}$$

again, as expected. □

Example 7.10 Let us now consider the Riemannian metric for a particle of mass m moving in a plane. The configuration manifold is $Q = \mathbb{R}^2$. If we use Cartesian coordinates, the kinetic energy is $\text{KE} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$, so giving the kinetic energy metric as

$$\mathbb{G} = m dx \otimes dx + m dy \otimes dy.$$

We wish to compute the representation of this Riemannian metric in polar coordinates. To do so, we use the change of basis formula for the basis vectors used in representing a Riemannian metric. One can also use the change of coordinates formula for components, but the basis vector method is often the most “natural.” In any case, we compute

$$\begin{aligned} \mathbb{G} &= m dx \otimes dx + m dy \otimes dy \\ &= m \left(\frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta \right) \otimes \left(\frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta \right) \\ &\quad + m \left(\frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta \right) \otimes \left(\frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta \right) \\ &= m (\cos \theta dr - r \sin \theta d\theta) \otimes (\cos \theta dr - r \sin \theta d\theta) \\ &\quad + m (\sin \theta dr + r \cos \theta d\theta) \otimes (\sin \theta dr + r \cos \theta d\theta) \\ &= m \cos^2 \theta dr \otimes dr + mr^2 \sin^2 \theta d\theta \otimes d\theta \end{aligned}$$

$$\begin{aligned} &\quad - mr \sin \theta \cos \theta (dr \otimes d\theta + d\theta \otimes dr) \\ &\quad + m \sin^2 \theta dr \otimes dr + mr^2 \cos^2 \theta d\theta \otimes d\theta \\ &\quad + mr \sin \theta \cos \theta (dr \otimes d\theta + d\theta \otimes dr) \\ &= m dr \otimes dr + mr^2 d\theta \otimes d\theta. \end{aligned}$$

It is important to note that the tensor product \otimes *is* linear with respect to scalar multiplication, but is *not* commutative. Thus, for example, $dr \otimes d\theta \neq d\theta \otimes dr$. \square

- Now we are ready to actually do something.

8. The Euler–Lagrange equations

- We now turn, at last, to the following important question:
Question: Given an interconnected mechanical system with configuration manifold Q and kinetic energy metric \mathbb{G} , what are the equations of motion for the system?
- *Answers:*
 1. The Newton–Euler equations, i.e., $F = ma$ and $M = J\dot{\omega}$.
 2. The “Euler–Lagrange equations.”
- We assume the reader knows about the first answer, we will come to understand the second answer, but we will (unfortunately) not come to understand why the two answers are equivalent.
- But we first make a small diversion to understand the natural context for the Euler–Lagrange equations.

8.1. The setup for the Euler–Lagrange equations

- We shall say a few words about the calculus of variations. We shall not be overly precise here, since this is not really a course on the calculus of variations.

Problem 8.1 Let $L: TQ \rightarrow \mathbb{R}$ be a C^∞ -function (called a *Lagrangian*), let $q_a, q_b \in Q$, and let $a < b \in \mathbb{R}$. We seek the curve $\gamma: [a, b] \rightarrow Q$ that minimizes the function A_L , defined on the set $C^2([a, b], q_a, q_b)$ of C^2 -curves defined on $[a, b]$, and satisfying $\gamma(a) = q_a$ and $\gamma(b) = q_b$, defined by

$$A_L(\gamma) = \int_a^b L(\gamma'(t)) dt. \quad \square$$

- The problem has the following solution.

Theorem 8.2 (Necessary conditions for minimization) *If a curve $\gamma \in C^2([a, b], q_a, q_b)$ minimizes A_L , then, for any chart (\mathcal{U}, ϕ) with the property that the image of γ intersects \mathcal{U} , the representation $t \mapsto \mathbf{q}(t)$ of γ in that chart must satisfy*

$$\frac{d}{dt} \left(\frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial q^i} = 0, \quad i \in \{1, \dots, n\},$$

at $(t, \mathbf{q}(t), \dot{\mathbf{q}}(t))$ for each t with the property that $\gamma(t) \in \mathcal{U}$.

- The equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial q^i} = 0, \quad i \in \{1, \dots, n\},$$

are the *Euler–Lagrange equations*.

- Although the form we give for the equations is *the* standard form, it is actually not clear. The clear version, for which the standard version is an abbreviation, is, using the Chain Rule,

$$\frac{\partial^2 L}{\partial v^i \partial v^j} \ddot{q}^j + \frac{\partial^2 L}{\partial v^i \partial q^j} \dot{q}^j - \frac{\partial L}{\partial q^i} = 0, \quad i \in \{1, \dots, n\}.$$

- The Euler–Lagrange equations are to Problem 8.1 what the condition the derivative equals zero is to the problem “find the minimum of a differentiable function $f: \mathbb{R}^n \rightarrow \mathbb{R}$.” That is to say, the Euler–Lagrange equations are a *necessary*, but *not sufficient*, condition for a solution of the minimization problem stated in Problem 8.1.

The Euler–Lagrange equations (cont’d)

- For an interconnected mechanical system with configuration manifold Q and kinetic energy metric \mathbb{G} , its *Lagrangian* is the function $L_{\mathbb{G}}: TQ \rightarrow \mathbb{R}$ defined by $L_{\mathbb{G}}(v_q) = \frac{1}{2} \mathbb{G}(v_q, v_q)$.
- The following theorem is the essential one connecting the Euler–Lagrange equations with Newtonian mechanics. Recall that $\Pi_a: Q \rightarrow \text{SO}(3) \times \mathbb{R}^3$ gives the position of the a th body at a configuration in Q .

Theorem 8.3 Consider an interconnected mechanical system comprised of rigid bodies $(\mathcal{B}_1, \mu_1), \dots, (\mathcal{B}_k, \mu_k)$, having configuration space Q , and having kinetic energy metric \mathbb{G} . For a curve $\gamma: I \rightarrow Q$ the following are equivalent:

- (i) the motion of the bodies, $t \mapsto \Pi_a \circ \gamma(t)$, satisfies the Newton–Euler equations;
- (ii) γ satisfies the Euler–Lagrange equations for the Lagrangian $L_{\mathbb{G}}$.

- We therefore need to understand the Euler–Lagrange equations for Lagrangians of the form $L_{\mathbb{G}}$. The most direct (and ugly) way to do this is by a coordinate calculation. See the text for slicker versions.
- We choose coordinates (q^1, \dots, q^n) , and write the Lagrangian in these coordinates as $L_{\mathbb{G}} = \frac{1}{2} \mathbb{G}_{jk} v^j v^k$. We then grind:

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L_{\mathbb{G}}}{\partial \dot{q}^i} \right) - \frac{\partial L_{\mathbb{G}}}{\partial q^i} &= \frac{d}{dt} (\mathbb{G}_{ij} \dot{q}^j) - \frac{1}{2} \frac{\partial \mathbb{G}_{jk}}{\partial q^i} \dot{q}^j \dot{q}^k \\ &= \mathbb{G}_{ij} \ddot{q}^j + \frac{\partial \mathbb{G}_{ij}}{\partial q^k} \dot{q}^j \dot{q}^k - \frac{1}{2} \frac{\partial \mathbb{G}_{jk}}{\partial q^i} \dot{q}^j \dot{q}^k. \end{aligned}$$

Let us denote

$$A_{ijk} = \frac{\partial \mathbb{G}_{ij}}{\partial q^k} - \frac{1}{2} \frac{\partial \mathbb{G}_{jk}}{\partial q^i},$$

and work on the term $A_{ijk} \dot{q}^j \dot{q}^k$. We first write

$$A_{ijk} = \underbrace{\frac{1}{2}(A_{ijk} + A_{ikj})}_{A_{ijk}^+} + \underbrace{\frac{1}{2}(A_{ijk} - A_{ikj})}_{A_{ijk}^-}.$$

Note that $A_{ijk}^- = -A_{ikj}^-$. Therefore,

$$A_{ijk}^- \dot{q}^j \dot{q}^k = -A_{ikj}^- \dot{q}^j \dot{q}^k = -A_{ijk}^- \dot{q}^j \dot{q}^k,$$

where the last step is just a renaming of the summation index. Thus $A_{ijk}^- \dot{q}^j \dot{q}^k = 0$ and so $A_{ijk} \dot{q}^j \dot{q}^k = A_{ijk}^+ \dot{q}^j \dot{q}^k$. We then compute

$$\begin{aligned} A_{ijk}^+ &= \frac{1}{2} \frac{\partial \mathbb{G}_{ij}}{\partial q^k} - \frac{1}{4} \frac{\partial \mathbb{G}_{jk}}{\partial q^i} + \frac{1}{2} \frac{\partial \mathbb{G}_{ik}}{\partial q^j} - \frac{1}{4} \frac{\partial \mathbb{G}_{kj}}{\partial q^i} \\ &= \frac{1}{2} \left(\frac{\partial \mathbb{G}_{ij}}{\partial q^k} + \frac{\partial \mathbb{G}_{ik}}{\partial q^j} - \frac{\partial \mathbb{G}_{jk}}{\partial q^i} \right). \end{aligned}$$

The Euler–Lagrange equations now look like

$$\mathbb{G}_{ij} \ddot{q}^j + \frac{1}{2} \left(\frac{\partial \mathbb{G}_{ij}}{\partial q^k} + \frac{\partial \mathbb{G}_{ik}}{\partial q^j} - \frac{\partial \mathbb{G}_{jk}}{\partial q^i} \right) \dot{q}^j \dot{q}^k = 0.$$

To get these in their final form, we multiply this equation by $\mathbb{G}^{\ell i}$, where we recall that $\mathbb{G}^{\ell i}$ are the components of the inverse of the matrix with components \mathbb{G}_{ij} .

We then have

$$\ddot{q}^{\ell} + \underbrace{\frac{1}{2} \mathbb{G}^{\ell i} \left(\frac{\partial \mathbb{G}_{ij}}{\partial q^k} + \frac{\partial \mathbb{G}_{ik}}{\partial q^j} - \frac{\partial \mathbb{G}_{jk}}{\partial q^i} \right)}_{\mathbb{G}^{\ell}_{jk}} \dot{q}^j \dot{q}^k = 0.$$

- The punchline is thus:

$$\begin{aligned} \ddot{q}^i + \overset{\mathbb{G}}{\Gamma}_{jk}^i \dot{q}^j \dot{q}^k &= 0, & i \in \{1, \dots, n\} \\ \overset{\mathbb{G}}{\Gamma}_{jk}^i &= \frac{1}{2} \overset{\mathbb{G}}{G}^{i\ell} \left(\frac{\partial \overset{\mathbb{G}}{G}_{\ell j}}{\partial q^k} + \frac{\partial \overset{\mathbb{G}}{G}_{\ell k}}{\partial q^j} - \frac{\partial \overset{\mathbb{G}}{G}_{jk}}{\partial q^\ell} \right), & i, j, k \in \{1, \dots, n\}. \end{aligned}$$

- There are (at least) two important things to note about these equations:
 1. We have spent significant time understanding what it takes to compute \mathbb{G} . Indeed, in some sense, much of the course to this point has been understanding what \mathbb{G} *is*, mathematically and physically, and understanding how to compute it. What we have seen is that, once you have the forward kinematic maps $\Pi_a : \mathbb{Q} \rightarrow \text{SO}(3) \times \mathbb{R}^3$, and the inertia tensors of the bodies, the computation of \mathbb{G} is algorithmic. Thus the computation of the equations of motion is also algorithmic, since they come to us in simple terms of the metric \mathbb{G} .
 2. The equations $\ddot{q}^i + \overset{\mathbb{G}}{\Gamma}_{jk}^i \dot{q}^j \dot{q}^k = 0$ are “the geodesic equations for the Levi-Civita affine connection associated to \mathbb{G} .”
- We now have to understand the words in the last statement.

9. Affine connections

- To define the notion of an affine connection, we need the following simple idea.

Definition 9.1 If X is a vector field on \mathbb{Q} and if f is a function on \mathbb{Q} , then the *Lie derivative* of f with respect to X is the function $\mathcal{L}_X f$ on \mathbb{Q} defined by

$$\mathcal{L}_X f(q) = \langle df(q); X(q) \rangle. \quad \square$$

- In coordinates we have

$$\begin{aligned} df &= \frac{\partial f}{\partial q^i} dq^i, & X &= X^i \frac{\partial}{\partial q^i} \\ \implies \mathcal{L}_X f &= X^i \frac{\partial f}{\partial q^i}. \end{aligned}$$

- This might be familiar to you as “the directional derivative of f with respect to X .”
- *Interpretation:* $\mathcal{L}_X f = 0$ if and only if f is constant along integral curves of X .

Affine connections (cont'd)

- We may now define what we mean by an affine connection. The definition seems strange, and the subsequent development will seem like it is going nowhere. The thing to keep in mind is that we are trying to explain the equation

$$\ddot{q}^i + \Gamma_{jk}^i \dot{q}^j \dot{q}^k = 0.$$

- Before we begin, it is useful to recall that if $X \in \Gamma^\infty(\text{TQ})$ and if $f \in C^\infty(\text{Q})$, then fX is a vector field, and it is defined by $(fX)(q) = f(q)X(q)$.

Definition 9.2 An *affine connection* on a manifold Q assigns to vector fields X and Y on Q a vector field $\nabla_X Y$, called the *covariant derivative* of Y with respect to X , and the assignment must satisfy the following rules:

- (i) the map $(X, Y) \mapsto \nabla_X Y$ is \mathbb{R} -bilinear;
- (ii) $\nabla_{fX} Y = f(\nabla_X Y)$ for $X, Y \in \Gamma^\infty(\text{TQ})$ and $f \in C^\infty(\text{Q})$;
- (iii) $\nabla_X fY = f(\nabla_X Y) + (\mathcal{L}_X f)Y$ for $X, Y \in \Gamma^\infty(\text{TQ})$ and $f \in C^\infty(\text{Q})$. □

- To get some idea of what an affine connection *is*, let us consider trying to represent a given affine connection ∇ in coordinates (q^1, \dots, q^n) .
- Note that $\nabla_{\frac{\partial}{\partial q^j}} \frac{\partial}{\partial q^k}$ is, by definition, a vector field. Therefore, for some functions Γ_{jk}^i , $i \in \{1, \dots, n\}$, we can write

$$\nabla_{\frac{\partial}{\partial q^j}} \frac{\partial}{\partial q^k} = \Gamma_{jk}^i \frac{\partial}{\partial q^i}.$$

- This then *defines* n^3 functions Γ_{jk}^i , $i, j, k \in \{1, \dots, n\}$, called the *Christoffel symbols* for ∇ in the given coordinates.
- We wish to show that the affine connection is uniquely determined in a given set of coordinates by its Christoffel symbols. To show this, consider vector fields $X = X^i \frac{\partial}{\partial q^i}$ and $Y = Y^j \frac{\partial}{\partial q^j}$, and compute

$$\begin{aligned} \nabla_X Y &= \nabla_{X^j \frac{\partial}{\partial q^j}} Y^k \frac{\partial}{\partial q^k} \\ &= X^j Y^k \nabla_{\frac{\partial}{\partial q^j}} \frac{\partial}{\partial q^k} + X^j (\mathcal{L}_{\frac{\partial}{\partial q^j}} Y^k) \frac{\partial}{\partial q^k} \\ &= \left(\frac{\partial Y^i}{\partial q^j} X^j + \Gamma_{jk}^i X^j Y^k \right) \frac{\partial}{\partial q^i}. \end{aligned}$$

- In the preceding computation, we have simply applied the defining properties of an affine connection. A good way to understand what an affine connection does is to understand the preceding computation. Notwithstanding that. . .
- *Punchline:* An affine connection is uniquely determined in a set of coordinates by its Christoffel symbols.
- Pushing on, let us define the covariant derivative in a different context. Given a curve $\gamma: I \rightarrow Q$, a *vector field along γ* is a map $Y: I \rightarrow TQ$ having the property that $Y(t) \in T_{\gamma(t)}Q$. Thus Y assigns a tangent vector to each point along the curve γ .
- Let Y be a vector field along a curve γ . We wish to define the covariant derivative of Y with respect to γ . We will soon see why we wish to do this.
- Since we only know how to compute the covariant derivative of a vector field with respect to another vector field, we need to convert our data of a vector field along a curve to something we can work with.
- We let X be a vector field for which γ is an integral curve. Thus $X(\gamma(t)) = \gamma'(t)$. We do not care about the values of X at points off γ .
- We let \bar{Y} be a vector field with the property that $\bar{Y}(\gamma(t)) = Y(t)$. We do not care about the values of \bar{Y} off γ .
- We now *define* the covariant derivative of Y with respect to γ to be the vector field $\nabla_{\gamma'(t)}Y(t)$ along γ defined by

$$\nabla_{\gamma'(t)}Y(t) = (\nabla_X\bar{Y})(\gamma(t)).$$

- One should check that this definition makes sense, in that it is independent of choices made for X and \bar{Y} . Also, one should check that X and \bar{Y} , having the desired properties, even exist. All this can be done.
- Let's see what this looks like in coordinates.

$$\begin{aligned} \nabla_{\gamma'(t)}Y(t) &= (\nabla_X\bar{Y})(\gamma(t)) \\ &= \left(\frac{\partial \bar{Y}^i}{\partial q^j} X^j + \Gamma_{jk}^i X^j \bar{Y}^k \right) \Big|_{q=\gamma(t)} \frac{\partial}{\partial q^i} \\ &= \left(\frac{\partial \bar{Y}^i}{\partial q^j} \dot{q}^j(t) + \Gamma_{jk}^i \dot{q}^j(t) \bar{Y}^k(t) \right) \frac{\partial}{\partial q^i} \\ &= (\dot{Y}^i(t) + \Gamma_{jk}^i \dot{q}^j(t) Y^k(t)) \frac{\partial}{\partial q^i}. \end{aligned}$$

In the last step, we used the Chain Rule.

- The special case when $Y(t) = \gamma'(t)$ is of particular interest.

Definition 9.3 A *geodesic* for an affine connection ∇ is a curve γ satisfying $\nabla_{\gamma'(t)}\gamma'(t) = 0$. □

- In coordinates, a geodesic simply second satisfies the second-order differential equation $\ddot{q}^i + \Gamma_{jk}^i \dot{q}^j \dot{q}^k = 0$, $i \in \{1, \dots, n\}$.
- This is beginning to really look like what we want, which you recall, is $\ddot{q}^i + \overset{\mathbb{G}}{\Gamma}_{jk}^i \dot{q}^j \dot{q}^k = 0$, $i \in \{1, \dots, n\}$.
- The question now boils down to, “Are the functions $\overset{\mathbb{G}}{\Gamma}_{jk}^i$, $i, j, k \in \{1, \dots, n\}$, the Christoffel symbols for some affine connection?”

Theorem 9.4 If \mathbb{G} is a Riemannian metric on \mathbb{Q} , then there exists a unique affine connection $\overset{\mathbb{G}}{\nabla}$, called the *Levi-Civita connection* for \mathbb{G} , for which the Christoffel symbols are exactly

$$\overset{\mathbb{G}}{\Gamma}_{jk}^i = \frac{1}{2} \overset{\mathbb{G}}{G}^{il} \left(\frac{\partial \overset{\mathbb{G}}{G}_{lj}}{\partial q^k} + \frac{\partial \overset{\mathbb{G}}{G}_{lk}}{\partial q^j} - \frac{\partial \overset{\mathbb{G}}{G}_{jk}}{\partial q^l} \right), \quad i, j, k \in \{1, \dots, n\}.$$

- There are more elegant characterisations of the Levi-Civita connection than we give. See the text for these.

Affine connections (cont'd)

- Let's see if we can do computations using affine connections.

Example 9.5 We take $Q = \mathbb{R}^2$, and consider a particle of mass $m = 1$ moving in this plane. The kinetic energy metric in Cartesian coordinates is

$$\mathbb{G} = dx \otimes dx + dy \otimes dy,$$

and in polar coordinates is

$$\mathbb{G} = dr \otimes dr + r^2 d\theta \otimes d\theta,$$

the latter from Example 7.10. Thus we have

$$\begin{aligned} \mathbb{G}_{xx} = \mathbb{G}_{yy} = 1, & \quad \mathbb{G}_{xy} = \mathbb{G}_{yx} = 0 \\ \mathbb{G}^{xx} = \mathbb{G}^{yy} = 1, & \quad \mathbb{G}^{xy} = \mathbb{G}^{yx} = 0, \end{aligned}$$

and

$$\begin{aligned} \mathbb{G}_{rr} = 1, \quad \mathbb{G}_{\theta\theta} = r^2, \quad \mathbb{G}_{r\theta} = \mathbb{G}_{\theta r} = 0 \\ \mathbb{G}^{rr} = 1, \quad \mathbb{G}^{\theta\theta} = \frac{1}{r^2}, \quad \mathbb{G}^{r\theta} = \mathbb{G}^{\theta r} = 0. \end{aligned}$$

We see that since the metric components are zero in Cartesian coordinates, the resulting Christoffel symbols are zero.

For polar coordinates, we won't go in detail through all the computations, but will only work out the nonzero Christoffel symbols. These turn out to be

$$\Gamma_{\theta\theta}^r = \frac{1}{2} \mathbb{G}^{rr} \left(\frac{\partial \mathbb{G}_{\theta r}}{\partial \theta} + \frac{\partial \mathbb{G}_{\theta r}}{\partial \theta} - \frac{\partial \mathbb{G}_{\theta\theta}}{\partial r} \right) + \frac{1}{2} \mathbb{G}^{r\theta} \left(\frac{\partial \mathbb{G}_{\theta\theta}}{\partial \theta} + \frac{\partial \mathbb{G}_{\theta\theta}}{\partial \theta} - \frac{\partial \mathbb{G}_{\theta\theta}}{\partial \theta} \right) = -r.$$

and

$$\Gamma_{r\theta}^{\theta} = \Gamma_{\theta r}^{\theta} = \frac{1}{2} \mathbb{G}^{\theta r} \left(\frac{\partial \mathbb{G}_{rr}}{\partial \theta} + \frac{\partial \mathbb{G}_{\theta r}}{\partial r} - \frac{\partial \mathbb{G}_{r\theta}}{\partial r} \right) + \frac{1}{2} \mathbb{G}^{\theta\theta} \left(\frac{\partial \mathbb{G}_{r\theta}}{\partial \theta} + \frac{\partial \mathbb{G}_{\theta\theta}}{\partial r} - \frac{\partial \mathbb{G}_{r\theta}}{\partial \theta} \right) = \frac{1}{r}.$$

With these computations, we easily compute the geodesic equations to be

$$\ddot{x} = 0, \quad \ddot{y} = 0$$

in Cartesian coordinates, and

$$\ddot{r} - r\dot{\theta}^2 = 0, \quad \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} = 0.$$

Note that geodesics are straight lines, and that the differential equations for straight lines are complicated in polar coordinates. \square

- Now we offer some physical interpretations of the geodesic equations, or, more generally, the expression $\nabla_{\gamma'(t)}\gamma'(t)$.

Example 9.6 (Geometric acceleration)

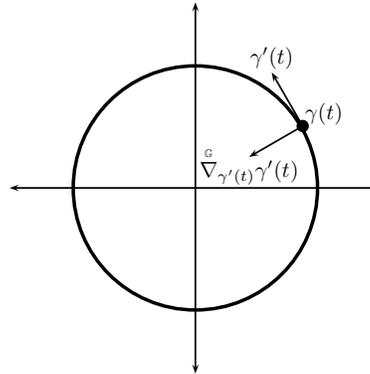
We claim that the quantity $\nabla_{\gamma'(t)}\gamma'(t)$ has a natural interpretation as acceleration.

Let us justify this with an example.

Consider the curve $\gamma: \mathbb{R} \rightarrow \mathbb{R}^2$ defined by $\gamma(t) = (\cos t, \sin t)$. This is a circle. The velocity along this curve has the usual expression as

$\gamma'(t) = ((\cos t, \sin t), (-\sin t, \cos t))$, and is as shown in the figure. This agrees with the usual

notion of velocity. What about acceleration? A naïve computation would be " $\gamma''(t) = ((\cos t, \sin t), (-\cos t, -\sin t))$ ". This makes sense to us, since we were taught that the acceleration along a circular trajectory of constant velocity should point toward the center of the circle.



In order for this definition of acceleration to make sense, it should be the same in any set of coordinates. Let us, therefore, represent the situation in polar coordinates. The

local representative of γ in polar coordinates is $t \mapsto (1, t)$. The local representative for the velocity γ' is then $t \mapsto ((1, t), (0, 1))$. This agrees with our Cartesian coordinate calculation, since this is simply a tangent vector of unit length tangent to the circle. How about acceleration. Well, our naïve computation would give the local representative of " $\gamma''(t)$ " as $t \mapsto ((1, t), (0, 0))$. Thus the naïve acceleration is zero in polar coordinates, but nonzero in Cartesian coordinates. Clearly this is inconsistent. The conclusion we must draw is that this naïve acceleration, obtained by simply twice differentiating coordinates with respect to time, is actually senseless.

We claim that this is all made better by instead thinking of acceleration as $\nabla_{\gamma'(t)}\gamma'(t)$, where we take ∇ to be the Levi-Civita connection associated with the standard Riemannian metric on \mathbb{R}^2 . To see this, we determine that the local representative of $\nabla_{\gamma'(t)}\gamma'(t)$ in Cartesian coordinates is $t \mapsto ((\cos t, \sin t), (-\cos t, -\sin t))$, i.e., the same as the naïve acceleration, since the Christoffel symbols are zero in Cartesian coordinates. In polar coordinates we compute the local representative of $\nabla_{\gamma'(t)}\gamma'(t)$ to be $t \mapsto ((1, t), (1, 0))$. Now this agrees with what we have in Cartesian coordinates!

Punchline: Be careful when you say that acceleration is the second derivative of position with respect to time! □

- Another physical interpretation of geodesics, at least for Levi-Civita connections, involves minimization of length.
- Let \mathbb{G} be a Riemannian metric on Q and let $\gamma: [a, b] \rightarrow Q$ be a curve. The *length* of γ is given by

$$\ell_{\mathbb{G}}(\gamma) = \int_a^b \sqrt{\mathbb{G}(\gamma'(t), \gamma'(t))} dt.$$

- Given $q_1, q_2 \in Q$ define the *Riemannian distance* between q_1 and q_2 by

$$d_{\mathbb{G}}(q_1, q_2) = \inf\{\ell_{\mathbb{G}}(\gamma) \mid \gamma: [0, 1] \rightarrow Q \text{ is piecewise } C^1, \text{ and } \gamma(0) = q_1 \text{ and } \gamma(1) = q_2\}.$$

- We then have the following characterization of curves that minimize distance.

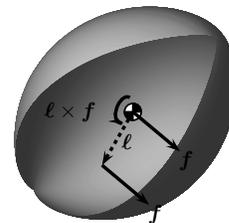
Theorem 9.7 (Distance minimizing properties of geodesics) *Let \mathbb{G} be a Riemannian metric on Q . If $\gamma: [a, b] \rightarrow Q$ is a piecewise C^1 -curve having the property that $\ell_{\mathbb{G}}(\gamma) = d_{\mathbb{G}}(q_1, q_2)$, then γ is a geodesic for $\nabla_{\mathbb{G}}$.*

10. Force

- We wish to do to the concept of force that which we have done to all other mechanical concepts: we wish to “geometrize” it.
- To do this, we should first come to a clear understanding of what a force is in Newtonian mechanics.

10.1. Forces and torques in Newtonian mechanics

- We consider a single rigid body (\mathcal{B}, μ) , which we suppose to be undergoing a motion specified by $t \mapsto (\mathbf{R}(t), \mathbf{r}(t))$.
- We consider a force \mathbf{f} applied at some point in the body, and a pure torque $\boldsymbol{\tau}$.
- The idea of torque as a vector should be interpreted thusly. The torque is of magnitude $\|\boldsymbol{\tau}\|_{\mathbb{R}^3}$ and applied about the axis $\boldsymbol{\tau}$ using the right-hand rule.
- A force \mathbf{f} applied at some point in the body is equivalent to the same force \mathbf{f} applied at the center of mass, along with a torque $\boldsymbol{\ell} \times \mathbf{f}$, where $\boldsymbol{\ell}$ is the vector from the center of mass to the point in the body where the force is applied (see figure).



- Therefore, without loss of generality, we assume that forces are applied at the center of mass of the body.
- We also suppose that forces are written relative to the spatial frame, not the body frame.
- Forces and torques give rise to the notion of work and power. We suppose that the force \mathbf{f} and the torque $\boldsymbol{\tau}$ are functions of time, and we denote by $\boldsymbol{\omega}$ the spatial angular velocity of the body at time t .
- The *translational work* done by \mathbf{f} along the motion is

$$W_{\text{tran}}(\mathbf{f}) = \int_I \langle \langle \mathbf{f}(t), \dot{\mathbf{r}}(t) \rangle \rangle_{\mathbb{R}^3} dt,$$

and the *rotational work* done by $\boldsymbol{\tau}$ along the motion is defined by

$$W_{\text{rot}}(\boldsymbol{\tau}) = \int_I \langle \langle \boldsymbol{\tau}(t), \boldsymbol{\omega}(t) \rangle \rangle_{\mathbb{R}^3} dt,$$

where $I \subset \mathbb{R}$ is the interval on which the motion is defined.

- We are also interested in power, which, you will recall, is the time derivative of work.
- Thus, the *translational power* and the *rotational power* for the force \mathbf{f} and the torque $\boldsymbol{\tau}$ by

$$P_{\text{tran}}(\mathbf{f})(t) = \langle \langle \mathbf{f}(t), \dot{\mathbf{r}}(t) \rangle \rangle_{\mathbb{R}^3}, \quad P_{\text{rot}}(\boldsymbol{\tau})(t) = \langle \langle \boldsymbol{\tau}(t), \boldsymbol{\omega}(t) \rangle \rangle_{\mathbb{R}^3},$$

respectively.

10.2. Force in Lagrangian mechanics

- We use the idea of power as our “in” to defining force for an interconnected mechanical system comprised of bodies $(\mathcal{B}_1, \mu_1), \dots, (\mathcal{B}_k, \mu_k)$ with configuration manifold \mathcal{Q} .
- A motion of the system is specified by a curve $\gamma: I \rightarrow \mathcal{Q}$, and this gives rise to motions $t \mapsto (\mathbf{R}_a(t), \mathbf{r}_a(t)) = \Pi_a \circ \gamma(t) \in \text{SO}(3) \times \mathbb{R}^3$ of each of the k bodies, where Π_a are the forward kinematic maps.

- We compute

$$\frac{d}{dt}(\Pi_a \circ \gamma(t)) = T_{\gamma(t)}\Pi_a(\gamma'(t)) = (\dot{\mathbf{R}}_a(t), \dot{\mathbf{r}}_a(t)),$$

where you recall that $T\Pi_a: TQ \rightarrow T(\text{SO}(3) \times \mathbb{R}^3)$ is the derivative (i.e., that which in coordinates is the Jacobian of Π_a).

- Now we define $\text{pr}_{\text{tran}}: T(\text{SO}(3) \times \mathbb{R}^3) \rightarrow \mathbb{R}^3$ and $\text{pr}_{\text{rot}}: T(\text{SO}(3) \times \mathbb{R}^3) \rightarrow \mathbb{R}^3$ are defined by

$$\text{pr}_{\text{tran}}(\mathbf{R}, \mathbf{r}, \dot{\mathbf{R}}, \dot{\mathbf{r}}) = \dot{\mathbf{r}}, \quad \text{pr}_{\text{rot}}(\mathbf{R}, \mathbf{r}, \dot{\mathbf{R}}, \dot{\mathbf{r}}) = (\dot{\mathbf{R}}\mathbf{R}^T)^\vee,$$

where \cdot^\vee is the inverse of the map $\hat{\cdot}$, which, you recall, converts a skew-symmetric 3×3 matrix into an element of \mathbb{R}^3 .

- Thus $\text{pr}_{\text{tran}} \circ T\Pi_a(\gamma'(t))$ is the velocity of the center of mass of the a th body as we undergo the motion defined by γ , and $\text{pr}_{\text{rot}} \circ T\Pi_a(\gamma'(t))$ is the spatial angular velocity of the a th body.

- For each $a \in \{1, \dots, k\}$, we note that the expression

$$\mathbb{G}_{\mathbb{R}^3}(\mathbf{f}_a(t), \text{pr}_{\text{tran}} \circ T\Pi_a(\gamma'(t))) + \mathbb{G}_{\mathbb{R}^3}(\boldsymbol{\tau}_a(t), \text{pr}_{\text{rot}} \circ T\Pi_a(\gamma'(t))) \quad (10)$$

is a linear function of $\gamma'(t) \in T_{\gamma(t)}Q$.

- Linear functions on $T_{\gamma(t)}Q$ are, by definition, elements of $T_{\gamma(t)}^*Q$. Therefore, (10) *defines* an element $F_{\mathbf{f}_a, \boldsymbol{\tau}_a}(t)$ of $T_{\gamma(t)}^*Q$.
- The *total external force* at time t is then the element $F(t) \in T_{\gamma(t)}^*Q$ given by $F(t) = \sum_{a=1}^k F_{\mathbf{f}_a, \boldsymbol{\tau}_a}(t)$.
- We will see how this rather abstract construction works in practice when we get to the examples.

10.3. Euler–Lagrange equations with forces

- Now that we know what a force *is* in Lagrangian mechanics, we need to understand how it comes into the equations of motion.

- We understand this in the Newtonian world, or at least we are supposed to. Let's make sure we do.
- Consider a single rigid body (\mathcal{B}, μ) undergoing a motion $t \mapsto (\mathbf{R}(t), \mathbf{r}(t))$ and subject to a force \mathbf{f} applied to the center of mass and a pure torque $\boldsymbol{\tau}$.
- The *spatial linear momentum* is $m\dot{\mathbf{r}}(t)$ and the *spatial angular momentum* is $\mathbb{I}_c(\boldsymbol{\omega}(t))$, where \mathbb{I}_c is the inertia tensor about the center of mass, and $\boldsymbol{\omega}(t) = \dot{\mathbf{R}}(t)\mathbf{R}^T(t)$ is the spatial angular velocity.
- The motion of the system is prescribed by the *Newton–Euler equations*, which we have alluded to, but not stated. They are these:

$$\frac{d}{dt}(\mathbf{p}(t)) = \mathbf{f}(t), \quad \frac{d}{dt}(\boldsymbol{\mu}(t)) = \boldsymbol{\tau}(t).$$

- Note that, if the external force and torque are zero, then the spatial linear and angular momenta are constant along the motion.

- Now we need to see how the Newton–Euler equations translate into the Lagrangian framework.

Theorem 10.1 *Let $(\mathcal{B}_1, \mu_1), \dots, (\mathcal{B}_k, \mu_k)$ be an interconnected mechanical system with configuration manifold \mathcal{Q} , and let \mathbb{G} be the kinetic energy Riemannian metric. Suppose that the a th body is subject to an external force \mathbf{f}_a applied at the center of mass, and to a pure torque $\boldsymbol{\tau}_a$, and let F be the corresponding total external force. For a curve γ on \mathcal{Q} , the following statements are equivalent:*

- (i) *the curve $\Pi_a \circ \gamma$ on $\text{SO}(3) \times \mathbb{R}^3$ describes a motion of the a th rigid body, $a \in \{1, \dots, k\}$, for the system according to the Newton–Euler equations;*
- (ii) $\overset{\mathbb{G}}{\nabla}_{\gamma'(t)}\gamma'(t) = \mathbb{G}^\# \circ F(\gamma'(t)).$

- The way to interpret the equation $\overset{\mathbb{G}}{\nabla}_{\gamma'(t)}\gamma'(t) = \mathbb{G}^\# \circ F(\gamma'(t))$ is “acceleration = mass⁻¹ × force,” analogously with “force = mass × acceleration” as per Newton.
- This makes sense because (1) we have argued that $\overset{\mathbb{G}}{\nabla}_{\gamma'(t)}\gamma'(t)$ can be interpreted as acceleration, and (2) since \mathbb{G} stands represents the inertial properties of the

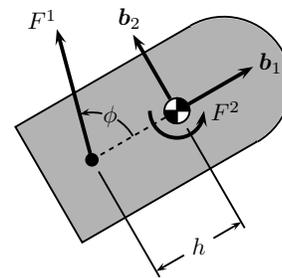
system, and since the components of \mathbb{G}^\sharp are the inverse of the components of \mathbb{G} , \mathbb{G}^\sharp can be thought of as mass^{-1} .

Force (cont'd)

- Let us see how the preceding discussion is handled in examples.
- In each example, we will produce the equations of motion $\mathbb{G} \nabla_{\gamma'(t)} \gamma'(t) = \mathbb{G}^\sharp \circ F(\gamma'(t))$, which in coordinates are

$$\ddot{q}^i + \overset{\mathbb{G}}{\Gamma}_{jk}^i \dot{q}^j \dot{q}^k = \overset{\mathbb{G}^\sharp}{F}^i, \quad i \in \{1, \dots, n\}. \quad (11)$$

Example 10.2 (Planar rigid body (cont'd)) The system has configuration manifold $Q = \mathbb{S}^1 \times \mathbb{R}^2$, and we denote by (θ, x, y) the coordinates we have been using all along. We consider a force $F(\cos(\theta + \phi), \sin(\theta + \phi), 0)$ applied at a point a distance h from the center of mass along the body b_1 -axis, and a pure torque $\tau(0, 0, 1)$. The force gives rise to a force $F(\cos(\theta + \phi), \sin(\theta + \phi), 0)$ applied at the center of mass and a torque $F(0, 0, -h \sin \phi)$. We denote the Lagrangian versions of these, which we wish to compute, by F^1 and F^2 . As with much of what we do, the forward kinematic map is key. We have previously



computed this map, in our coordinates, to have the local representative

$$(x, y, \theta) \mapsto \left(\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, (x, y, 0) \right).$$

We may then easily compute the spatial angular velocity to be $\omega = \dot{\theta}(0, 0, 1)$. The coordinate expression for the power generated by the force and the torque, acting together on a tangent vector v_q with components $(\dot{x}, \dot{y}, \dot{\theta})$, is then computed to be

$$F \cos(\theta + \phi)\dot{x} + F \sin(\theta + \phi)\dot{y} - Fh \sin \phi \dot{\theta} + \tau \dot{\theta},$$

using the formulae for translational and rotational power. We now note that this is the same as $F^1(v_q) + F^2(v_q)$, where

$$F^1 = F(\cos(\theta + \phi)dx + \sin(\theta + \phi)dy - h \sin \phi d\theta), \quad F^2 = \tau d\theta.$$

The total external force is then $F = F^1 + F^2$, which gives a covector field on Q . The components of F might depend on time, position, or velocity.

Now let us determine the equations of motion, which we know to be

$\overset{\mathbb{G}}{\nabla}_{\gamma'(t)} \gamma'(t) = \mathbb{G}^\sharp \circ F(\gamma'(t))$. We have

$$\mathbb{G} = m dx \otimes dx + m dy \otimes dy + J d\theta \otimes d\theta.$$

It is now just a matter of doing calculations. Since the components of the kinetic energy metric are constant in our chosen coordinates, the Christoffel symbols are zero. The matrix representative for \mathbb{G} are

$$[\mathbb{G}] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{bmatrix},$$

meaning that the matrix representative of \mathbb{G}^\sharp is

$$[\mathbb{G}^\sharp] = \begin{bmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{1}{J} \end{bmatrix}.$$

The components of F are

$$(F \cos(\theta + \phi), F \sin(\theta + \phi), \tau - Fh \sin \phi).$$

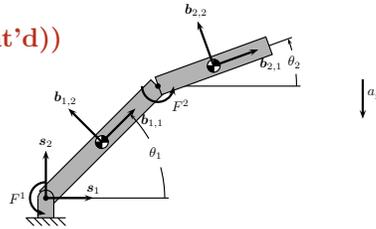
Thus the equations of motion, following (11), are simply

$$\ddot{x} = \frac{1}{m}F \cos(\theta + \sin \phi), \quad \ddot{y} = \frac{1}{m}F \sin(\theta + \sin \phi), \quad \ddot{\theta} = \frac{1}{J}(\tau - Fh \sin \phi).$$

The way we are thinking of things at the moment, the coefficients F and τ of the force and the torque are given, and we are charged with computing the resulting motion of the body. When we come to talk about control theory, we will be interested in designing F and τ to produce a motion with desired properties. \square

Example 10.3 (Two-link manipulator (cont'd))

The system has configuration manifold $Q = \mathbb{S}^1 \times \mathbb{S}^1$ with coordinates (θ_1, θ_2) . We consider a torque $\tau_1(0, 0, 1)$ applied to link 1, corresponding to a torque provided by a motor at the base of the link. We also have a torque applied to the connection between the links. This will apply a torque $\tau_2(0, 0, 1)$ to link 2 and a torque $-\tau_2(0, 0, 1)$ to the second link. The corresponding Lagrangian forces we denote by F^1 and F^2 , respectively. Again, the forward kinematic maps are key. We had computed these to be have local representatives



$$(\theta_1, \theta_2) \mapsto (\mathbf{R}_a, \mathbf{r}_a), \quad a \in \{1, 2\},$$

where

$$\mathbf{R}_a = \begin{bmatrix} \cos \theta_a & -\sin \theta_a & 0 \\ \sin \theta_a & \cos \theta_a & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad a \in \{1, 2\},$$

and where

$$\mathbf{r}_1 = (\frac{1}{2}l_1 \cos \theta_1, \frac{1}{2}l_1 \sin \theta_1, 0),$$

$$\mathbf{r}_2 = (l_1 \cos \theta_1 + \frac{1}{2}l_2 \cos \theta_2, l_1 \sin \theta_1 + \frac{1}{2}l_2 \sin \theta_2, 0).$$

The coordinate representations for the spatial angular velocities of the two links are easily computed to be $\omega_1 = \dot{\theta}_1(0, 0, 1)$ and $\omega_2 = \dot{\theta}_2(0, 0, 1)$. Now one applies the definition of translational power (which is zero in this case) and rotational power to get the expression $\tau_1\dot{\theta}_1 + \tau_2(\dot{\theta}_2 - \dot{\theta}_1)$ for the power supplied to the system by the forces. This is equivalent to $F^1(v_q) + F^2(v_q)$, where

$$F^1 = \tau_1 d\theta_1, \quad F^2 = \tau_2(d\theta_2 - d\theta_1),$$

and where $v_q = (\dot{\theta}_1, \dot{\theta}_2)$. The total external force is then $F = F^1 + F^2$.

Next, we compute the equations of motion, using (11). We had computed the kinetic energy metric to have the coordinate representation

$$\begin{aligned} & (J_1 + \frac{1}{4}(m_1 + 4m_2)\ell_1^2)d\theta_1 \otimes d\theta_1 + \frac{1}{2}m_2\ell_1\ell_2 \cos(\theta_1 - \theta_2)d\theta_1 \otimes d\theta_2 \\ & + \frac{1}{2}m_2\ell_1\ell_2 \cos(\theta_1 - \theta_2)d\theta_2 \otimes d\theta_1 + (J_2 + \frac{1}{4}m_2\ell_2^2)d\theta_2 \otimes d\theta_2. \end{aligned}$$

Now there is no nice way to do the calculations in (11). The Christoffel symbols are ugly, the components of \mathbb{G}^\sharp are ugly. It's just ugly, and this is not even a complicated example. But the advantage to what we are doing is that the process can be automated by a symbolic manipulation program. The equations with no external forces are given in the text, and the equations with external forces are straightforward, but lengthy, modifications of these. \square

10.4. Forces, generally speaking

- The preceding considerations indicate how one can convert Newtonian formulations of force into Lagrangian formulations of force.
- Let us provide the general definition of a force in Lagrangian mechanics, so we can see how it is abstracted geometrically.

Definition 10.4 A *force* on a configuration manifold Q is a map $F: \mathbb{R} \times TQ \rightarrow T^*Q$ with the property that $F(t, v_q) \in T_q^*Q$. \square

- Thus a force assigns to each time, position, and velocity, a force, by which it is meant a cotangent vector at that configuration. Forces depending on time might be things like user-supplied external force. Commonly encountered forces that are dependent on velocity are viscous dissipation forces, which are proportional to velocity.
- Forces that do not depend on time are called *time-independent*.

10.5. Potential forces

- A very special sort of force often arises in applications, and this is the potential force.
- A *potential force* is one of the form $F(t, v_q) = -dV(q)$, for a function V on Q , called the *potential energy function*.

- Potential forces have certain interesting properties.

Proposition 10.5 *If F is a force that is time-independent, the following statements are equivalent:*

- (i) F is a potential force;
- (ii) for any C^0 -curve $\gamma: [0, T] \rightarrow \mathbb{Q}$ having the property that $\gamma(0) = \gamma(T)$,

$$\int_0^T \langle F(\gamma'(t)); \gamma'(t) \rangle dt = 0.$$

- The way to read the second part of the proposition is: “ F does no work on closed curves.”

Example 10.6 Consider a planar single-link robot with one end fixed to ground, and a motor that supplies a unit torque to the link. That $\mathbb{Q} = \mathbb{S}^1$ and we denote by θ the usual angular coordinate. The torque in the Newtonian setting is $(0, 0, 1)$, and in the Lagrangian framework is $F = d\theta$. This seems like a potential force, since in coordinates it is the differential of the function θ . However, consider the closed curve γ defined by $[0, 2\pi] \ni t \mapsto (\cos t, \sin t) \in \mathbb{Q}$. We compute

$$\int_0^{2\pi} \langle F(\gamma'(t)); \gamma'(t) \rangle dt = \int_0^{2\pi} \langle d\theta; \frac{\partial}{\partial \theta} \rangle dt = 2\pi.$$

Thus the work done around this closed curve is not zero! So this is not a potential force. The seeming contradiction is resolved by noting that θ is not a continuous function on \mathbb{Q} . □

- A common potential force is the gravitational force exerted on a body. In the Newtonian setting, this force has magnitude ma_g , where m is the mass of the body and a_g is the acceleration due to gravity, and is applied to the center of mass in the direction of the gravitational field.

Example 10.7 (Two-link manipulator (cont'd)) We suppose that gravity acts in the direction of $-s_2$. Thus the Newtonian representation of the gravitational force on link a is $-m_a a_g(0, 1, 0)$. The power exerted by the gravitational force on the a th link is then $-m_a a_g \dot{y}_a(t)$, where y_a is the velocity of the center of mass of the a th link. Thus y_a is the second component of the vector r_a describing the position of the center of mass of link a . Let us think of y_a as being a function on \mathbb{Q} . The force on \mathbb{Q} corresponding to the gravitational force is

$$F_g = -m_1 a_g dy_1 - m_2 a_g dy_2.$$

Indeed, with F_g defined in this way, we have

$$\langle F_g; \gamma'(t) \rangle = -m_1 a_g dy_1(\gamma'(t)) - m_2 a_g dy_2(\gamma'(t)) = -m_1 a_g \dot{y}_1 - m_2 a_g \dot{y}_2,$$

which is the desired expression for the Lagrangian force. Moreover, it is clear that $F_g = -dV$ where $V = m_1 a_g y_1 + m_2 a_g y_2$. Thus F_g is a potential force. Given expressions for y_1 and y_2 as previously computed, we have

$$V(\theta_1, \theta_2) = \frac{1}{2} m_1 a_g \ell_1 \sin \theta_1 + m_2 a_g \ell_1 \sin \theta_1 + \frac{1}{2} m_2 \ell_2 \sin \theta_2.$$

and

$$F_g = -\left(\frac{1}{2} m_1 \ell_1 a_g \cos \theta_1 + m_2 \ell_1 a_g \cos \theta_1\right) d\theta_1 - \frac{1}{2} m_2 \ell_2 a_g \cos \theta_2 d\theta_2. \quad \square$$

11. An introduction to control theory for mechanical systems

- We now turn over a new page, and talk about control theory.

Definition 11.1 A *simple mechanical control system* is a 4-tuple $(Q, \mathbb{G}, V, \mathcal{F} = \{F^1, \dots, F^m\})$, where

- (i) Q is the configuration manifold,
- (ii) \mathbb{G} is the kinetic energy Riemannian metric,
- (iii) V is the potential function, and
- (iv) F^1, \dots, F^m are covector fields on Q called *input forces*. □

- The equations governing a simple mechanical control system are

$$\underbrace{\overset{\mathbb{G}}{\nabla}_{\gamma'(t)} \gamma'(t)}_{\text{uncontrolled dynamics}} = -\mathbb{G}^\sharp \circ dV(\gamma(t)) + \underbrace{\sum_{a=1}^m u_a(t) \mathbb{G}^\sharp \circ F^a(\gamma(t))}_{\text{control force}}. \quad (12)$$

- The idea is that if we do not provide a control force, then the system will evolve corresponding to the Riemannian metric \mathbb{G} and potential function V .
- However, if we do provide control forces, then there are various sorts of control theoretic problems we can talk about. Here are some.
 1. *Stabilization*: Given a configuration q_0 , we wish to design the controls u_1, \dots, u_m as functions of state (i.e., position and velocity) so that the system with these controls possesses q_0 as a stable equilibrium point. If you have had an introductory course in linear system theory, this is probably the problem you studied most of the time.
 2. *Trajectory tracking*: Given a reference trajectory $\gamma_{\text{ref}}: I \rightarrow \mathbb{Q}$, find controls that follow the reference trajectory as closely as possible.
 3. *Controllability*: Given configurations $q_1, q_2 \in \mathbb{Q}$, does there exist controls $u_a: [0, T] \rightarrow \mathbb{R}$, $a \in \{1, \dots, m\}$, such that, if at time 0 the system is at rest at q_1 , then at time T the system is at rest at q_2 . Again, those having an introductory linear systems course will have encountered controllability for linear systems. As we shall see, controllability for nonlinear systems is much harder.
 4. *Motion planning*: Here, one wishes not only to assert the existence of controls steering the system from q_1 to q_2 , as in the controllability problem, but to *find* these controls.

11.1. Linearization of simple mechanical control systems

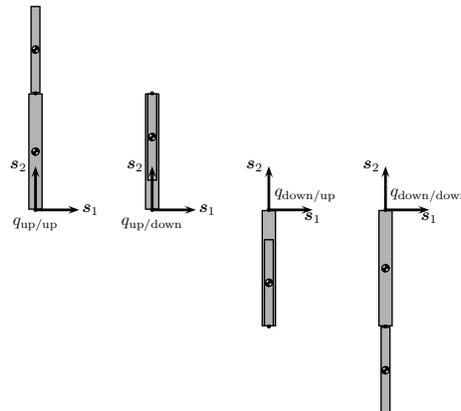
- Let us consider, for a moment, the stabilization problem. If one wishes to stabilize the configuration q_0 , the first thing one normally does is linearize, and check to see if the linearization is controllable.
- First we need the notion of an equilibrium configuration. Recall that an equilibrium configuration is one where, if the system starts there, it stays there.

Definition 11.2 A *controlled equilibrium configuration* for a simple mechanical control system $(Q, \mathbb{G}, V, \mathcal{F} = \{F^1, \dots, F^m\})$ is a pair $(q_0, u_0) \in Q \times \mathbb{R}^m$ with the property that

$$dV(q_0) = \sum_{a=1}^m u_{0,a} F^a(q_0). \quad \square$$

- Note that (q_0, u_0) is a controlled equilibrium configuration if and only if $t \mapsto (q_0, u_0)$ satisfies (12).
- Note that $(q_0, 0)$ is a controlled equilibrium configuration if and only if $dV(q_0) = 0$.

Example 11.3 (Two-link manipulator (cont'd)) Let us consider our two-link manipulator example with coordinates (θ_1, θ_2) . We suppose that the system has an input torque only at the base of the first link. Let us see what the controlled equilibria are. First we look for controlled equilibria of the form $(q_0, 0)$. One can check that the points q_0 that satisfy $dV(q_0) = 0$ have coordinate values $(\frac{\pi}{2}, \frac{\pi}{2}), (\frac{\pi}{2}, -\frac{\pi}{2}), (-\frac{\pi}{2}, \frac{\pi}{2}), (-\frac{\pi}{2}, -\frac{\pi}{2})$. These are depicted as below.



These could also have been guessed on physical grounds, I suppose. If we allow

equilibria with nonzero control, then, provided that the control force u_0 can be chosen sufficiently large, any configuration having coordinate values $(\theta_{1,0}, \frac{\pi}{2})$ or $(\theta_{1,0}, -\frac{\pi}{2})$ is possible, where $\theta_{1,0}$ is arbitrary. \square

- Let us simplify matters in the sequel by assuming that we are interested only in controlled equilibrium configurations of the form $(q_0, 0)$, i.e., those for which $dV(q_0) = 0$.
- Next we wish to linearise about such an equilibrium.
- To see how to do this, let us write (12) in coordinates, after multiplying the equation by \mathbb{G}^b :

$$\mathbb{G}_{\ell i} \ddot{q}^i + \mathbb{G}_{\ell i} \overset{\mathbb{G}}{\Gamma}_{jk}^i \dot{q}^j \dot{q}^k = -\frac{\partial V}{\partial q^\ell} + \sum_{a=1}^m u_a F_\ell^a. \quad (13)$$

- To linearize, we take, in coordinates, $q(t) = q_0 + \theta(t)$ and $u(t) = u + \mu(t)$. Thus $(q_0, 0)$ is the controlled equilibrium configuration, and $(\theta(t), \mu(t))$ is the deviation of the configuration and the control from the equilibrium.

- To do linearize in a careful way, we would substitute these expressions for $q(t)$ and $\mu(t)$ into (13), and then Taylor expand about $(q_0, 0)$. Let us make some observations that allow us to record the results of these computations without actually having to do them:

1. the term $\mathbb{G}_{\ell i}(q(t)) \ddot{q}^i(t)$ is linear in $\ddot{q}(t)$, and so will yield $\mathbb{G}_{\ell i}(q_0) \ddot{\theta}^i(t)$ upon linearization;
2. the term $\mathbb{G}_{\ell i}(q(t)) \overset{\mathbb{G}}{\Gamma}_{jk}^i(q(t)) \dot{q}^j \dot{q}^k$ is quadratic in $\dot{q}(t)$, and so will vanish upon linearization;
3. the term $-\frac{\partial V}{\partial q^\ell}(q(t))$ is linearized in the standard manner, and yields $\frac{\partial^2 V}{\partial q^\ell \partial q^i}(q_0) \theta^i$ upon linearization;
4. the term $\sum_{a=1}^m u_a F_\ell^a(q(t))$ is linear in u , and so yields $\sum_{a=1}^m u_a F_\ell^a(q_0)$ upon linearization.

- By understanding the above comments, you will have demonstrated a pretty good understanding of the process of linearization.

- Recall that the symmetric matrix with components $\frac{\partial^2 V}{\partial q^\ell \partial q^i}(q_0)$ is called the **Hessian** of V at q_0 . It is actually the matrix representation of a symmetric bilinear map on $T_{q_0}\mathbb{Q}$, and this map we denote by $\text{Hess } V(q_0)$.
- We have thus demonstrated that the linearization in coordinates is

$$\mathbb{G}_{\ell i}(q_0)\ddot{\theta}^i(t) + \frac{\partial^2 V}{\partial q^\ell \partial q^i}\theta^i(t) = \sum_{a=1}^m u_a F_\ell^a(q_0).$$

- In coordinate independent notation, the linearization has the form

$$M^b(\ddot{x}(t)) + K^b(x(t)) = F(u(t)), \quad (14)$$

where

- $t \mapsto x(t)$ is a curve in $T_{q_0}\mathbb{Q}$,
- $M = \mathbb{G}(q_0)$,
- $K = \text{Hess } V(q_0)$,
- and $F \in L(\mathbb{R}^m; T_{q_0}^*\mathbb{Q})$ is defined by $F(u) = \sum_{a=1}^m u_a F_\ell^a(q_0)$.

- Punchline:** After linearization of (12) at a controlled equilibrium $(q_0, 0)$, we arrive at a second-order linear equation on $T_{q_0}\mathbb{Q}$ given by (14).
- Now let us analyze (14).
- In stabilization theory using linearization, one requires that the linearization be controllable (or more generally, stabilizable, but let's stick with the slightly less general requirement of controllable).
- So we should check the controllability of the linear system (14).
- First we convert it to a first-order system so we can use the standard theory of controllability. We have

$$\begin{aligned} \ddot{x}(t) + M^\sharp \circ K^b(x(t)) &= M^\sharp \circ F(u(t)) \\ \implies \begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & \text{id}_V \\ -M^\sharp \circ K^b & 0 \end{bmatrix}}_A \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ M^\sharp \circ F \end{bmatrix}}_B u. \end{aligned}$$

- The Kalman Rank Condition says that the linear system (14) is controllable if and only if the linear map $\left[B \mid AB \mid \dots \mid A^{2n-1}B \right]$ has maximal rank.
- A straightforward computation yields

$$\begin{aligned} & \left[B \mid AB \mid \dots \mid A^{2n-1}B \right] \\ &= \begin{bmatrix} 0 & B_0 & 0 & A_0 \circ B_0 & \dots & 0 & A_0^{n-1} \circ B_0 \\ B_0 & 0 & A_0 \circ B_0 & 0 & \dots & A_0^{n-1} \circ B_0 & 0 \end{bmatrix}, \end{aligned}$$

where $A_0 = -M^\sharp \circ K^\flat$ and $B_0 = M^\sharp \circ F$.

- This directly gives the following result.

Theorem 11.4 *The linear system (14) is controllable if and only if the rank of the linear map*

$$\left[B_0 \mid A_0 B_0 \mid \dots \mid A_0^{n-1} B_0 \right]$$

is maximal, where $A_0 = -M^\sharp \circ K^\flat$ and $B_0 = M^\sharp \circ F$.

- There is an interesting special case, that when $V = 0$. This is not uncommon in applications.

Corollary 11.5 *If $(\mathbb{Q}, \mathbb{G}, V = 0, \mathcal{F} = \{F^1, \dots, F^m\})$ is a simple mechanical control system with zero potential, then*

- (i) *for any $q_0 \in \mathbb{Q}$, $(q_0, 0)$ is a controlled equilibrium configuration and*
- (ii) *the linearization at a controlled equilibrium configuration $(q_0, 0)$ is controllable if and only if $\text{span}_{\mathbb{R}} \{F^1(q_0), \dots, F^m(q_0)\} = \mathbb{T}_{q_0}^* \mathbb{Q}$, i.e., if and only if the system is fully actuated.*

- If $V = 0$ and the system is not fully actuated, then stabilization of controlled equilibria $(q_0, 0)$ is hard. For example, for such a system, it is not possible to design the controls to be continuous functions of the state (i.e., of position and velocity) that will render q_0 asymptotically stable.
- In fact, nothing you have encountered in your control courses to this point will enable you to stabilize such equilibria.

12. Motion planning for affine connection control systems

- We consider simple mechanical control systems $(Q, \mathbb{G}, V = 0, \mathcal{F} = \{F^1, \dots, F^m\})$ with zero potential, which is then governed by the equations

$$\overset{\mathbb{G}}{\nabla}_{\gamma'(t)} \gamma'(t) = \sum_{a=1}^m u_a(t) \mathbb{G}^\sharp \circ F^a(\gamma(t)). \quad (15)$$

- In actuality, there is no reason why we cannot consider a generalization of (15). In particular, we can consider a general affine connection ∇ and, rather than forces F^1, \dots, F^m , consider vector fields Y_1, \dots, Y_m .

Definition 12.1 An *affine connection control system* is a triple $(Q, \nabla, \mathcal{Y} = \{Y_1, \dots, Y_m\})$ where

- (i) Q is a manifold,
- (ii) ∇ is an affine connection on Q ,
- (iii) Y_1, \dots, Y_m are vector fields on Q . □

- The governing equations are

$$\nabla_{\gamma'(t)} \gamma'(t) = \sum_{a=1}^m u^a(t) Y_a(\gamma(t)). \quad (16)$$

- The special case in (15) occurs when $\nabla = \overset{\mathbb{G}}{\nabla}$ and $Y_a = \mathbb{G}^\sharp \circ F^a$, $a \in \{1, \dots, m\}$.
- In this course, we will not encounter physical systems for which the extra generality is useful. However, there are such systems, one being the rolling disk that we have encountered previously.
- However, it is no more difficult to consider the general case, and you can keep the special case of (15) in mind if it is comforting.

12.1. The controllability problem for affine connection control systems

- We shall briefly consider the general controllability problem for an affine connection control system $\Sigma = (Q, \nabla, \mathcal{Y})$.

- A *controlled trajectory* for Σ is a pair (γ, u) where $u: [0, T] \rightarrow \mathbb{R}^m$ is Lebesgue integrable, and where $\gamma: [0, T] \rightarrow Q$ is such that (16) is satisfied.
- Define

$$\mathcal{R}_{\Sigma, Q}(q_0, T) = \{ \gamma(T) \mid (\gamma, u) \text{ is a controlled trajectory and } \gamma'(0) = 0_{q_0} \},$$

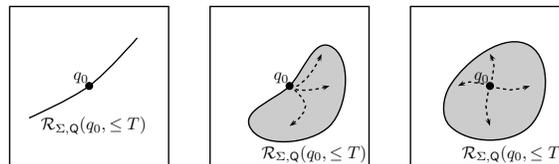
$$\mathcal{R}_{\Sigma, Q}(q_0, \leq T) = \bigcup_{t \in [0, T]} \mathcal{R}_{\Sigma, Q}(q_0, t).$$

- Thus $\mathcal{R}_{\Sigma, Q}(q_0, T)$ is the set of configurations reachable from q_0 in time T starting with zero initial velocity, and $\mathcal{R}_{\Sigma, Q}(q_0, \leq T)$ is set of configurations reachable from q_0 in time at most T starting with zero initial velocity.

Definition 12.2 Σ is

- (i) *configuration accessible* if there exists $T > 0$ such that $\text{int}(\mathcal{R}_{\Sigma, Q}(q_0, \leq t)) \neq \emptyset$ for $t \in]0, T]$, and is
- (ii) is *small-time locally configuration controllable (STLCC)* from q_0 if there exists $T > 0$ such that $q_0 \in \text{int}(\mathcal{R}_{\Sigma, Q}(q_0, \leq t))$ for $t \in]0, T]$. \square

- Here's a pictorial representation of the distinction between configuration accessibility and configuration controllability:



On the left, the system is not configuration accessible since the reachable set has empty interior, in the middle the system is configuration accessible since the reachable set has nonempty interior, but is not configuration controllable since q_0 is not in the interior of the reachable set, on the right, the system is configuration controllable.

- We will not say much in general about configuration accessibility or configuration controllability. Let us content ourselves with this.
 1. Configuration accessibility of affine connection control systems is comparatively easy; see Theorem 7.36.
 2. Configuration controllability is hard. Very hard, in fact. See Section 7.3.3 and Chapter 8 in the text for some results.

- We shall sidestep this difficult problem, and directly look at what seems like a harder problem, the motion planning problem.
- It turns out that, for affine connection control systems, there is a class of systems, containing many interesting physical examples, for which the motion planning problem has a comparatively easy solution.
- This requires introducing a simpler class of problems that, on the surface, have no relationship with affine connection control systems.

12.2. Driftless systems

Definition 12.3 A *driftless system* is a pair $(Q, \mathcal{X} = \{X_1, \dots, X_m\})$ where

- (i) Q is a manifold and
 - (ii) X_1, \dots, X_m are vector fields on Q . □
- The equations governing a driftless system are

$$\gamma'(t) = \sum_{a=1}^m \tilde{u}^a X_a(\gamma(t)). \quad (17)$$

- For a driftless system $\Sigma = (Q, \mathcal{X})$ we have controllability notions mirroring those we gave for affine connection control systems.
- A *controlled trajectory* for Σ is a pair (γ, u) where $u: [0, T] \rightarrow \mathbb{R}^m$ is Lebesgue integrable, and where $\gamma: [0, T] \rightarrow Q$ is such that (17) is satisfied.
- Define

$$\begin{aligned} \mathcal{R}_\Sigma(q_0, T) &= \{ \gamma(T) \mid (\gamma, u) \text{ is a controlled trajectory and } \gamma(0) = q_0 \}, \\ \mathcal{R}_\Sigma(q_0, \leq T) &= \bigcup_{t \in [0, T]} \mathcal{R}_\Sigma(q_0, t). \end{aligned}$$

- Thus $\mathcal{R}_\Sigma(q_0, T)$ is the set of configurations reachable from q_0 in time T , and $\mathcal{R}_\Sigma(q_0, \leq T)$ is set of configurations reachable from q_0 in time at most T .

Definition 12.4 Σ is

- (i) *accessible* if there exists $T > 0$ such that $\text{int}(\mathcal{R}_\Sigma(q_0, \leq t)) \neq \emptyset$ for $t \in]0, T]$, and is
- (ii) *small-time locally controllable (STLC)* from q_0 if there exists $T > 0$ such that $q_0 \in \text{int}(\mathcal{R}_\Sigma(q_0, \leq t))$ for $t \in]0, T]$. □

Motion planning for affine connection control systems (cont'd)

- Unlike affine connection control systems, the matter of controllability for driftless systems is comparatively easy.
- To understand this, we introduce a fun new tool.
- Recall that a vector field X can be used to differentiate a function via $\mathcal{L}_X f = X^i \frac{\partial f}{\partial q^i}$. The map $f \mapsto \mathcal{L}_X f$ is

1. \mathbb{R} -linear and
2. satisfies $\mathcal{L}_X(fg) = f\mathcal{L}_X g + g\mathcal{L}_X f$.

It is additionally true that, to any such operation on infinitely differentiable functions, there is a vector field for which the operation is Lie differentiation with respect to that vector field.

- Given vector fields X and Y , one can verify that

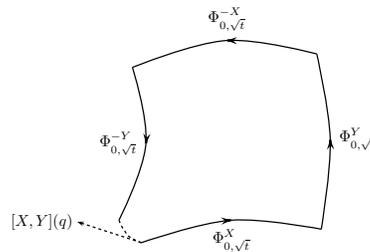
$$f \mapsto \mathcal{L}_X \mathcal{L}_Y f - \mathcal{L}_Y \mathcal{L}_X f$$

satisfies the properties above. Therefore, there is associated to this operation a vector field. We denote this vector field by $[X, Y]$, and call it the **Lie bracket** of X and Y .

- In coordinates,

$$[X, Y] = \left(\frac{\partial Y^i}{\partial q^j} X^j - \frac{\partial X^i}{\partial q^j} Y^j \right) \frac{\partial}{\partial q^i}.$$

- A telling characterization of the Lie bracket is the following.



Proposition 12.5 Let $X, Y \in \Gamma^\infty(\text{TQ})$ and let $q \in \text{Q}$. Define a curve γ at q by

$$\gamma(t) = \Phi_{\sqrt{t}}^{-Y} \circ \Phi_{\sqrt{t}}^{-X} \circ \Phi_{\sqrt{t}}^Y \circ \Phi_{\sqrt{t}}^X(q).$$

Then γ is differentiable and $\gamma'(0) = [X, Y](q)$.

Example 12.6 Take $\text{Q} = \mathbb{R}^3$, and $X = \frac{\partial}{\partial y}$ and $Y = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}$. We compute $[X, Y] = \frac{\partial}{\partial z}$. Let us see if we can illustrate Proposition 12.5 “by hand.” The

differential equations corresponding to the vector fields X and Y are

$$\begin{aligned} \dot{x} &= 0 & \dot{x} &= 1 \\ \dot{y} &= 1 & \text{and} & \dot{y} = 0 \\ \dot{z} &= 0 & \dot{z} &= y, \end{aligned}$$

respectively. Using these differential equations, one can readily compute

$$\begin{aligned} \Phi_{\sqrt{t}}^X(0, 0, 0) &= (0, \sqrt{t}, 0) \\ \Phi_{\sqrt{t}}^Y(0, \sqrt{t}, 0) &= (\sqrt{t}, \sqrt{t}, t) \\ \Phi_{\sqrt{t}}^{-X}(\sqrt{t}, \sqrt{t}, t) &= (\sqrt{t}, 0, t) \\ \Phi_{\sqrt{t}}^{-Y}(\sqrt{t}, 0, t) &= (0, 0, t). \end{aligned}$$

Thus

$$[X, Y](0, 0, 0) = \left. \frac{d}{dt} \right|_{t=0} \Phi_{\sqrt{t}}^{-Y} \circ \Phi_{\sqrt{t}}^{-X} \circ \Phi_{\sqrt{t}}^Y \circ \Phi_{\sqrt{t}}^X(0, 0, 0),$$

just as claimed. \square

- The Lie bracket tells us that by switching back and forth between flowing along X and Y , we can move in a direction that is possibly aligned with neither X and Y . (Think about parallel parking your car.)
- Let us see how this is related to driftless systems.
- We consider a two-input driftless system for simplicity:

$$\gamma'(t) = u^1(t)X_1(t) + u^2(t)X_2(t).$$

- Consider the following control defined on $[0, 4\sqrt{T}]$:

$$(u^1(t), u^2(t)) = \begin{cases} (1, 0), & t \in [0, \sqrt{T}[, \\ (0, 1), & t \in [\sqrt{T}, 2\sqrt{T}[, \\ (-1, 0), & t \in [2\sqrt{T}, 3\sqrt{T}[, \\ (0, -1), & t \in [3\sqrt{T}, 4\sqrt{T}]. \end{cases}$$

- Note that for the first \sqrt{T} seconds we follow the integral curve of X_1 , for the second \sqrt{T} seconds we follow the integral curve of X_2 , for the third \sqrt{T} seconds we follow the integral curve of $-X_1$, and for the fourth \sqrt{T} seconds we follow the integral curve of $-X_2$.

- Proposition 12.5 then says that at time $4\sqrt{T}$ the system will have moved, to lowest order, in the direction of $[X_1, X_2]$.
- This has clear implications in terms of controllability: points in the direction of $[X_1, X_2](q)$ should lie in the reachable set.
- One can show (less directly than above) that points in the direction of $[X_1, [X_1, X_2]](q)$ and $[X_2, [X_1, X_2]](q)$ also lie in the reachable set.
- Most generally, one can show that, for a driftless system $(Q, \mathcal{X} = \{X_1, \dots, X_m\})$, any point lying in the direction of an arbitrary iterated Lie bracket lies in the reachable set.
- For this reason we define $\text{Lie}^{(\infty)}(\mathcal{X})_q$ to be the subspace of $T_q Q$ generated by all iterated Lie brackets of the vector fields X_1, \dots, X_m .
- A useful computational fact is that it is sufficient to consider brackets of the form $[X_{a_1}, [X_{a_2}, \dots, [X_{a_{k-1}}, X_{a_k}]]]$.

- The following theorem characterize controllability for driftless systems.

Theorem 12.7 For a driftless system $\Sigma = (Q, \mathcal{X})$, the following statements are equivalent:

- (i) Σ is accessible from q_0 ;
- (ii) Σ is STLC from q_0 ;
- (iii) $\text{Lie}^{(\infty)}(\mathcal{X})_{q_0} = T_{q_0} Q$.

Example 12.8 Take $Q = \mathbb{R}^3$, and $X_1 = \frac{\partial}{\partial y}$ and $X_2 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}$. We had computed $[X_1, X_2] = \frac{\partial}{\partial z}$. Thus, for every $q = (x, y, z) \in Q$, we have $\text{Lie}^{(\infty)}(\mathcal{X})_q = T_q Q$. Therefore, the driftless system is STLC from every point in Q .

Motion planning for affine connection control systems (cont'd)

12.3. Reducing motion planning for affine connection control systems to motion planning for driftless systems

- We have seen that controllability for driftless systems, controllability is easier than it is for affine connection control systems.
- The same assertion holds for motion planning. The motion planning problem for driftless systems is not trivial, but it is a problem that possible is solvable.
- Therefore, if we can reduce the motion planning problem for a given affine connection control systems to one for a driftless system, we will have accomplished something.

Problem 12.9 Given an affine connection control system

$\Sigma_{\text{dyn}} = (\mathbb{Q}, \nabla, \mathcal{Y} = \{Y_1, \dots, Y_m\})$ and a driftless system

$\Sigma_{\text{kin}} = (\mathbb{Q}, \mathcal{X} = \{X_1, \dots, X_{\tilde{m}}\})$, are there relationships between the controlled trajectories:

$$\nabla_{\gamma'(t)} \gamma'(t) = \sum_{a=1}^m u_{\text{dyn}}^a(t) Y_a(\gamma(t)) \Leftarrow ? \Rightarrow \gamma'(t) = \sum_{\alpha=1}^{\tilde{m}} u_{\text{kin}}^\alpha X_\alpha(\gamma(t)).$$

- Note that typically there will be no useful relationships between an affine connection control system and a driftless system.
- There are some restrictions that immediately apply. Let us make some observations along this line. To do so, for each $q \in \mathbb{Q}$, define subspaces \mathcal{Y}_q and \mathcal{X}_q of $T_q \mathbb{Q}$ by

$$\mathcal{Y}_q = \text{span}_{\mathbb{R}} \{Y_1(q), \dots, Y_m(q)\}, \quad \mathcal{X}_q = \text{span}_{\mathbb{R}} \{X_1(q), \dots, X_{\tilde{m}}(q)\}.$$

1. If (γ, u_{kin}) is a controlled trajectory for Σ_{kin} , then $\gamma'(t) \in \mathcal{X}_{\gamma'(t)}$. Thus the only trajectories of Σ_{dyn} that can appear as trajectories of Σ_{kin} are those whose velocities lie in \mathcal{X} .
 2. The trajectories of Σ_{dyn} are smoother than those of Σ_{kin} .
- Let us illustrate this second point explicitly.

Example 12.10 We take the World's Simplest Example, a particle of mass m moving on a line, and subject to a control force. Thus $Q = \mathbb{R}$ (with coordinate (x)), $\mathbb{G} = m dx \otimes dx$, and $F^1 = dx$. The equation for the associated affine connection control system is

$$\ddot{x}(t) = \frac{u_{\text{dyn}}(t)}{m}.$$

Thus the input vector field is $Y_1 = \frac{1}{m} \frac{\partial}{\partial x}$. We take the driftless system $(Q, \{X_1\})$ with $X_1 = \frac{\partial}{\partial x}$. The equation for the driftless system is

$$\dot{x}(t) = u_{\text{kin}}(t).$$

Consider a controlled trajectory for Σ_{kin} with

$$u_{\text{kin}}(t) = \begin{cases} 1, & t \in [0, \frac{1}{2}], \\ -1, & t \in]\frac{1}{2}, 1]. \end{cases}$$

If $x(0) = 0$ then the resulting trajectory is

$$\gamma(t) = \begin{cases} t, & t \in [0, \frac{1}{2}], \\ 1 - t, & t \in]\frac{1}{2}, 1]. \end{cases}$$

Question: Is there a control u_{dyn} for Σ_{dyn} so that (γ, u_{dyn}) is a controlled trajectory for Σ_{dyn} ?

Answer: No, because γ is not sufficiently differentiable. At $t = \frac{1}{2}$, the velocity is discontinuous, and a discontinuity in velocity implies “infinite” acceleration, i.e., “infinite” force.

To overcome this difficulty, one can reparameterize γ so that $\gamma'(t) = 0$ at times t where γ' has a discontinuity. In doing so, the trajectory in Q will follow the same path, but will do so with a different speed than γ . However, the reparameterized path *will* be followable by a trajectory for the system with force as an input. \square

- Thus we need to ensure that the controls for Σ_{kin} are sufficiently nice that any controlled trajectories for Σ_{kin} are of a class that can be followed by controlled trajectories of Σ_{dyn} .
- We let \mathcal{U}_{kin} and \mathcal{U}_{dyn} be classes of inputs for driftless systems and affine connection control systems, respectively, with the property that controls in \mathcal{U}_{kin} are “one integration smoother” than controls in \mathcal{U}_{dyn} .
- For example, if \mathcal{U}_{dyn} consists of locally integrable controls, then \mathcal{U}_{kin} consists of locally absolutely continuous controls, if you know what those words mean.

12.4. Kinematically controllable systems

- We now consider more precisely a case when there is a relationship between an affine connection control system and a driftless system.
- If $\gamma: I \rightarrow Q$ is a differentiable curve on Q , a *reparameterization* of I is a map $\tau: J \rightarrow I$ where $J \subset \mathbb{R}$ is another interval and where $\tau'(t) > 0$ for all $t \in \text{int}(J)$. The corresponding *reparameterization* of γ is the curve $\gamma \circ \tau$.

Definition 12.11 For an affine connection control system $\Sigma_{\text{dyn}} = (Q, \nabla, \mathcal{Y})$, a *decoupling vector field* is a vector field X on Q having the property that, for every integral curve γ of X , and every reparameterization τ of γ , there exists a control $u_{\text{dyn}} \in \mathcal{U}_{\text{dyn}}$ for which $(\gamma \circ \tau, u_{\text{dyn}})$ is a controlled trajectory for Σ_{dyn} . \square

- The idea is that one can follow integral curves of a decoupling vector field, and do so speeding up and slowing down as desired.

Motion planning for affine connection control systems (cont'd)

- It is easy to check if a given vector field is a decoupling vector field.

Theorem 12.12 A vector field X on Q is a decoupling vector field for the affine connection control system (Q, ∇, \mathcal{Y}) if and only if $X(q) \in \mathcal{Y}_q$ and $\nabla_X X(q) \in \mathcal{Y}_q$ for each $q \in Q$.

- Note that if X is a decoupling vector field, then so is $-X$.
- It is not so easy to find decoupling vector fields, but there are some techniques that give one some guidance. We refer to Chapter 8 in the text.
- Let us suppose that we have decoupling vector fields X_1, \dots, X_m .
- Starting at $q_0 \in Q$, construct a curve in Q by following the integral curve for X_{a_1} for time t_1 , then following the integral curve for X_{a_2} for time t_2 , and so on, up to following the integral curve of X_{a_k} for time t_k .
- Now reparameterize this curve so that, on each segment, one starts and stops with zero velocity.

- One can then follow the entire curve, since at the points where we switch decoupling vector fields, the velocity is zero, and so we avoid discontinuities in velocity.
- This suggests the following strategy for motion planning.
 1. Find enough decoupling vector fields $X_1, \dots, X_{\bar{m}}$ so that the motion planning problem can be solved for the driftless system $(Q, \mathcal{X} = \{X_1, \dots, X_{\bar{m}}\})$.
 2. Suppose that the solution to the driftless motion planning comes in the form of a sequence of integral curves of the vector fields $X_1, \dots, X_{\bar{m}}$ as described above.
 3. Reparameterize each segment so that it starts and ends with zero velocity.
 4. On each segment, find controls for Σ_{dyn} that follow the integral curve for the driftless system.

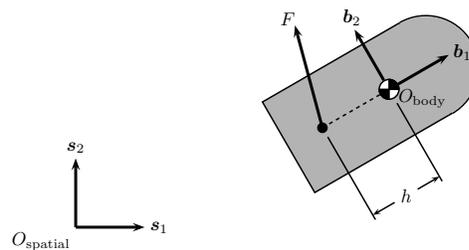
- Based on this strategy, we have the following definition.

Definition 12.13 An affine connection control system $\Sigma_{\text{dyn}} = (Q, \nabla, \mathcal{Y})$ is *kinematically controllable* if it possesses decoupling vector fields $\mathcal{X} = \{X_1, \dots, X_{\bar{m}}\}$ for which the driftless system $\Sigma_{\text{kin}} = (Q, \mathcal{X})$ is controllable. \square

Example 12.14 We consider the planar rigid body that we talked about back in the introductory lecture. The configuration space is $Q = \mathbb{S}^1 \times \mathbb{R}^2$, and we use coordinates (θ, x, y) . The kinetic energy metric has matrix representation

$$[\mathbb{G}] = \begin{bmatrix} J & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}.$$

The force is broken into two components, one along the body b_1 -axis, and one along the body b_2 -axis. These forces are, after going through the usual process of



conversion,

$$F^1 = \cos \theta dx + \sin \theta dy, \quad F^2 = -\frac{h}{J} d\theta - \sin \theta dx + \cos \theta dy.$$

Converting these to vector fields $Y_a = \mathbb{G}^\# \circ F^a$, $a \in \{1, 2\}$, gives

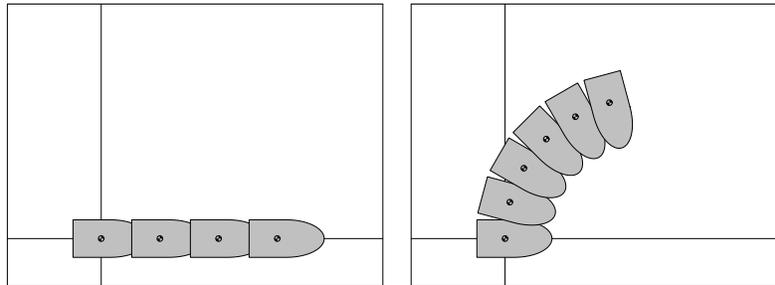
$$Y_1 = \frac{\cos \theta}{m} \frac{\partial}{\partial x} + \frac{\sin \theta}{m} \frac{\partial}{\partial y}, \quad Y_2 = -\frac{h}{J} \frac{\partial}{\partial \theta} - \frac{\sin \theta}{m} \frac{\partial}{\partial x} + \frac{\cos \theta}{m} \frac{\partial}{\partial y}.$$

Let us look for decoupling vector fields. There is no sure way of doing this (well, actually there is in this case, but never mind). We know that decoupling vector fields must lie in the span of Y_1 and Y_2 . So, for a lark, let us see if Y_1 and Y_2 are decoupling. We should check to see whether $\nabla_{Y_1} Y_1$ and/or $\nabla_{Y_2} Y_2$ lie in the span of Y_1 and Y_2 . Let us compute...

$$\nabla_{Y_1} Y_1 = 0, \quad \nabla_{Y_2} Y_2 = \frac{h \cos \theta}{mJ} \frac{\partial}{\partial x} + \frac{h \sin \theta}{mJ} \frac{\partial}{\partial y}.$$

Note that both of these vector fields are indeed in the span of Y_1 and Y_2 . Thus they are both decoupling vector fields.

Let us see what the motion of the body looks like along integral curves of these decoupling vector fields.



On the left is motion along Y_1 and on the right is motion along Y_2 . Can we do motion planning using these decoupling vector fields. We can eyeball the motions in the above figure, and believe that this might seem feasible. Or, we can check for kinematic controllability. Thus we should check whether the driftless system $(Q, \{Y_1, Y_2\})$ is controllable. We compute

$$[Y_1, Y_2] = -\frac{h \sin \theta}{mJ} \frac{\partial}{\partial x} + \frac{h \cos \theta}{mJ} \frac{\partial}{\partial y}.$$

One can readily verify that the three vector fields $\{Y_1, Y_2, [Y_1, Y_2]\}$ are linearly independent at each point in Q , so the system is indeed controllable. Thus $(Q, \nabla, \{Y_1, Y_2\})$ is kinematically controllable. \square

12.5. Maximally reducible systems

- We have seen that motion planning for affine connection control systems, using driftless systems, is possible, via the notion of kinematic controllability.
- However, kinematic controllability comes with a switching character, with lots of starting and stopping. Can we improve on this?

Definition 12.15 An affine connection control system $\Sigma_{\text{dyn}} = (\mathbb{Q}, \nabla, \mathcal{Y})$ is *maximally reducible* to a driftless system $\Sigma_{\text{kin}} = (\mathbb{Q}, \mathcal{X})$ if,

- (i) for every controlled trajectory (γ, u_{kin}) for Σ_{kin} , there exists a control u_{dyn} so that (γ, u_{dyn}) is a controlled trajectory for Σ_{dyn} , and if,
- (ii) for every controlled trajectory (γ, u_{dyn}) for Σ_{dyn} with the property that $\gamma'(0) \in \text{span}_{\mathbb{R}} \{X_1(\gamma(0)), \dots, X_{\bar{m}}(\gamma(0))\}$, there exists a control u_{kin} so that (γ, u_{kin}) is a controlled trajectory for Σ_{kin} . □

- The first condition is that every trajectory of the driftless system can be followed by a trajectory of the affine connection control system.

- The second condition must be interpreted more carefully. Note that it is impossible that every trajectory of the affine connection be followable with a trajectory of the driftless system. This is because the trajectories of the driftless system are restricted to lie in the span of the vector fields $\{X_1, \dots, X_{\bar{m}}\}$, while the velocities of the affine connection control system are unrestricted.
- Thus the second condition says, roughly, that every trajectory of the affine connection control system that can possibly be followed, can in actuality be followed by a trajectory of the driftless system.
- To characterize maximal reducibility, we introduce the *symmetric product* between vector fields X and Y :

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X.$$

Theorem 12.16 An affine connection control system $\Sigma_{\text{dyn}} = (\mathbb{Q}, \nabla, \mathcal{Y})$ is maximally reducible to a driftless system $\Sigma_{\text{kin}} = (\mathbb{Q}, \mathcal{X})$ if and only if

- (i) $\text{span}_{\mathbb{R}} \{Y_1(q), \dots, Y_m(q)\} = \text{span}_{\mathbb{R}} \{X_1(q), \dots, X_{\bar{m}}(q)\}$ for each $q \in \mathbb{Q}$, and
- (ii) $\langle Y_a : Y_b \rangle(q) \in \text{span}_{\mathbb{R}} \{Y_1(q), \dots, Y_m(q)\}$ for each $a, b \in \{1, \dots, m\}$ and $q \in \mathbb{Q}$.

- Note that an affine connection control system Σ_{dyn} is always maximally reducible to essentially the same driftless system, i.e., one whose input vector fields have a span equal to the span equal to the input vector fields for Σ_{dyn} . Thus, we can simply say that Σ_{dyn} is *maximally reducible* if it is reducible to *some* driftless system.

Example 12.17 Let us return to the planar body example, and check whether it is maximally reducible. Here are some calculations:

$$\begin{aligned}\langle Y_1 : Y_1 \rangle &= 0, & \langle Y_1 : Y_2 \rangle &= \frac{h \sin \theta}{mJ} \frac{\partial}{\partial x} - \frac{h \cos \theta}{mJ} \frac{\partial}{\partial y}, \\ \langle Y_2 : Y_2 \rangle &= \frac{2h \cos \theta}{mJ} \frac{\partial}{\partial x} + \frac{2h \sin \theta}{mJ} \frac{\partial}{\partial y}.\end{aligned}$$

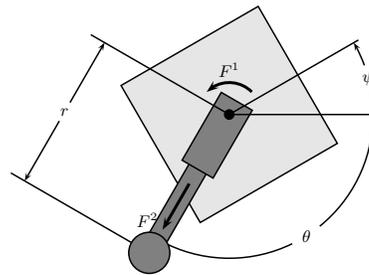
Note that while $\langle Y_1 : Y_1 \rangle$ and $\langle Y_2 : Y_2 \rangle$ lie in the span of the vector fields Y_1 and Y_2 (we have essentially already seen this with our decoupling vector field calculations, since $\langle X : X \rangle = 2\nabla_X X$), $\langle Y_1 : Y_2 \rangle$ does not. Therefore, this system is not maximally reducible. The consequence of this for motion planning is that, if we wish to use a driftless system as our basis for motion planning, we must live with the start/stop nature of the motion. \square

Example 12.18 Let us look at a new example, called “the robotic leg.” This system is pinned to ground, and consists of a body, on top of which sits an extensible leg with a mass on the tip. This mass can be moved in and out. There is also a torque that actuates the leg. Let us quickly summarize the data. We have $Q = \mathbb{R}_+ \times \mathbb{S}^1 \times \mathbb{S}^1$, coordinates (r, θ, ψ) , a kinetic energy metric with matrix representation

$$[\mathbb{G}] = \begin{bmatrix} m & 0 & 0 \\ 0 & mr^2 & 0 \\ 0 & 0 & J \end{bmatrix},$$

giving rise to the nonzero Christoffel symbols $\Gamma_{\theta\theta}^r = -r$ and $\Gamma_{r\theta}^\theta = \Gamma_{\theta r}^\theta = \frac{1}{r}$. The input forces are $F^1 = d\theta - d\psi$ and $F^2 = dr$, giving the input vector fields

$$Y_1 = \frac{1}{mr^2} \frac{\partial}{\partial \theta} - \frac{1}{J} \frac{\partial}{\partial \psi}, \quad Y_2 = \frac{1}{m} \frac{\partial}{\partial r}.$$



We compute

$$\langle Y_1 : Y_1 \rangle = -\frac{2}{m^2 r^3} \frac{\partial}{\partial r}, \quad \langle Y_1 : Y_2 \rangle = 0, \quad \langle Y_2 : Y_2 \rangle = 0.$$

We readily see that the system is maximally reducible. Furthermore,

$$[Y_1, Y_2] = \frac{2}{m^2 r^3} \frac{\partial}{\partial \theta},$$

from which we deduce that the associated driftless system is controllable. Note that maximal reducibility allows us to follow not just the vector fields Y_1 and Y_2 , but any linear combination of them. This allows greater flexibility in the design of control laws. \square

Definition 12.19 An affine connection $\Sigma_{\text{dyn}} = (Q, \nabla, \mathcal{Y})$ is *maximally reducibly kinematically controllable (MR-KC)* if it is maximally reducible, and if the driftless system $\Sigma_{\text{kin}} = (Q, \mathcal{Y})$ is controllable. \square