Minitutorial
Distributed Control and Coordination Algorithms

Francesco Bullo and Jorge Cortés

[1]Department of Mechanical Engineering
University of California, Santa Barbara
bullo@engineering.ucsb.edu

[2]Mechanical and Aerospace Engineering
University of California, San Diego
cortes@ucsd.edu

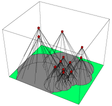
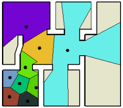
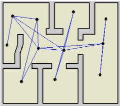
SIAM Conference on Dynamical Systems
Snowbird, Utah, May 17, 2009

Based on joint book with Sonia Martínez (UCSD)

Cooperative multi-agent systems

What kind of systems?
Groups of agents with control, sensing, communication and computing

Each individual
- **senses** its immediate environment
- **communicates** with others
- **processes** information gathered
- **takes local action** in response

Self-organized behaviors in biological groups




Decision making in animals

Able to
- deploy over a given region
- assume specified pattern
- rendezvous at a common point
- jointly initiate motion/change direction in a synchronized way

Species achieve synchronized behavior
- with limited sensing/communication between individuals
- without apparently following group leader

(Couzin et al, Nature 05; Conradt et al, Nature 03)

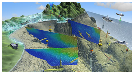
Engineered multi-agent systems

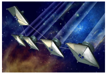
Embedded robotic systems and sensor networks for

- high-stress, rapid deployment — e.g., disaster recovery networks
- distributed environmental monitoring — e.g., portable chemical and biological sensor arrays detecting toxic pollutants
- autonomous sampling for biological applications — e.g., monitoring of species in risk, validation of climate and oceanographic models
- science imaging — e.g., multispacecraft distributed interferometers flying in formation to enable imaging at microarcsecond resolution


Sandia National Labs


UCSD Scripps


MBARI AOSN


NASA

Research challenges

What useful engineering tasks can be performed

with limited-sensing/communication agents?

Dynamics	simple interactions give rise to rich emerging behavior
Feedback	rather than open-loop computation for known/static setup
Information flow	who knows what, when, why, how, dynamically changing
Reliability/performance	robust, efficient, predictable behavior

How to coordinate individual agents into coherent whole?

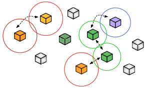
Objective: *systematic methodologies to design and analyze cooperative strategies to control multi-agent systems*

Research program: what are we after?

Design of provably correct coordination algorithms for basic tasks

Formal model to rigorously formalize, analyze, and compare coordination algorithms

Mathematical tools to study convergence, stability, and robustness of coordination algorithms



Coordination tasks

exploration, map building, search and rescue,
surveillance, odor localization, monitoring, distributed sensing

Technical approach

Optimization Methods

- resource allocation
- geometric optimization
- load balancing

Geometry & Analysis

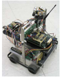
- computational structures
- differential geometry
- nonsmooth analysis

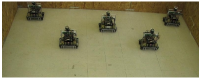
Control & Robotics

- algorithm design
- cooperative control
- stability theory

Distributed Algorithms

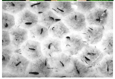
- adhoc networks
- decentralized vs centralized
- emerging behaviors





What we will see in this lecture

Basic motion coordination tasks:
get together at a point, stay connected, deploy over a region



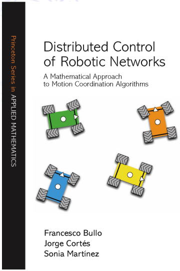
Design coordination algorithms that achieve these tasks and analyze their correctness and time complexity

Expand set of math tools: invariance principles for non-deterministic systems, geometric optimization, non-smooth stability analysis

Robustness against link failures, agents' arrivals and departures, delays, asynchronism

Image credits: jupiterimages and Animal Behavior

Text: Distributed Control of Robotic Networks



1. intro to distributed algorithms (graph theory, synchronous networks, and averaging algos)
2. geometric models and geometric optimization problems
3. model for robotic, relative sensing networks, and complexity
4. algorithms for rendezvous, deployment, boundary estimation

Status: Freely downloadable at
http://coordinationbook.info
with tutorial slides & software libraries.
Shortly on sale by Princeton Univ Press

Outline

Models for multi-agent networks

References

1. I. Suzuki and M. Yamashita. Distributed anonymous mobile robots: Formation of geometric patterns. *SIAM Journal on Computing*, 28(4):1347–1363, 1999
2. N. A. Lynch. *Distributed Algorithms*. Morgan Kaufmann, 1997
3. D. P. Bertsekas and J. N. Tsitsiklis. *Parallel and Distributed Computation: Numerical Methods*. Athena Scientific, 1997
4. S. Martínez, F. Bullo, J. Cortés, and E. Frazzoli. On synchronous robotic networks – Part I: Models, tasks and complexity. *IEEE Transactions on Automatic Control*, 52(12):2199–2213, 2007

Objective

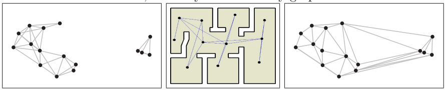
1. meaningful + tractable model
2. feasible operations and their cost
3. control/communication tradeoffs

Robotic network

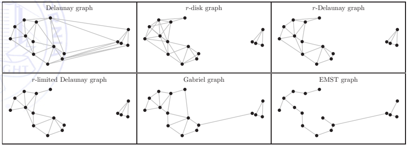
A **uniform/anonymous robotic network** $\mathcal{S}$ is

1. $I = \{1, \dots, N\}$; **set of unique identifiers (UIDs)**
2. $\mathcal{A} = \{A^{[i]}\}_{i \in I}$, with $A^{[i]} = (X, U, f)$ is a **set of physical agents**
3. **interaction graph**

Disk, visibility and Delauney graphs



Communication models for robotic networks



Relevant graphs

1. fixed, directed, balanced
2. switching
3. geometric or state-dependent
4. random, random geometric

Message model

1. message
2. packet/bits
3. absolute or relative positions
4. packet losses

Prototypical examples

Locally-connected first-order robots in $\mathbb{R}^d$ $\mathcal{S}_{\text{disk}}$

- n points $x^{[1]}, \dots, x^{[n]}$ in $\mathbb{R}^d$, $d \geq 1$
- obeying $\dot{x}^{[i]}(t) = u^{[i]}(t)$, with $u^{[i]} \in [-u_{\max}, u_{\max}]$
- identical robots of the form

$$(\mathbb{R}^d, [-u_{\max}, u_{\max}]^d, \mathbb{R}^d, (\mathbf{0}, \boldsymbol{e}_1, \dots, \boldsymbol{e}_d))$$

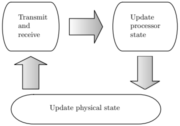
- each robot communicates to other robots within r

Variations

1. $\mathcal{S}_{\text{D}}$ same dynamics, but Delaunay graph
2. $\mathcal{S}_{\text{LD}}$: same dynamics, but r-limited Delaunay graph
3. $\mathcal{S}_{\text{vehicles}}$: same graph, but nonholonomic dynamics

Synchronous control and communication

1. **communication schedule** $\mathbb{T} = \{t_\ell\}_{\ell \in \mathbb{N}_0} \subset \mathbb{R}_{\geq 0}$
2. **communication alphabet** L including the `null` message
3. set of values for **logic variables** W
4. **message-generation function** $\text{msg}\colon \mathbb{T} \times X \times W \times I \to L$
5. **state-transition functions** $\text{stf}\colon \mathbb{T} \times W \times L^N \to W$
6. **control function** $\text{ctrl}\colon \mathbb{R}_{\geq 0} \times X \times W \times L^N \to U$



Task and complexity

- **Coordination task** is $(\mathcal{W}, \mathcal{T})$ where $\mathcal{T}\colon X^N \times \mathcal{W}^N \to \{\texttt{true}, \texttt{false}\}$

  Logic-based: achieve consensus, synchronize, form a team
  Motion: deploy, gather, flock, reach pattern
  Sensor-based: search, estimate, identify, track, map

- For $\{\mathcal{S}, \mathcal{T}, \mathcal{CC}\}$, define costs/complexity:

  control effort, communication packets, computational cost

- **Time complexity to achieve $\mathcal{T}$ with $\mathcal{CC}$**

$$\mathsf{TC}(\mathcal{T}, \mathcal{CC}, x_0, w_0) = \inf \, \{\ell \mid \mathcal{T}(x(t_k), w(t_k)) = \texttt{true}\,, \text{ for all } k \geq \ell\}$$

$$\mathsf{TC}(\mathcal{T}, \mathcal{CC}) = \sup \, \{\, \mathsf{TC}(\mathcal{T}, \mathcal{CC}, x_0, w_0) \mid (x_0, w_0) \in X^N \times \mathcal{W}^N \}$$

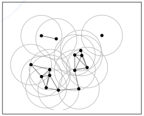
$$\mathsf{TC}(\mathcal{T}) = \inf \, \{\, \mathsf{TC}(\mathcal{T}, \mathcal{CC}) \mid \mathcal{CC} \text{ achieves } \mathcal{T}\}$$

Outline

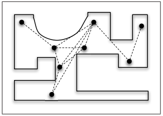
1. Models for multi-agent networks
2. Rendezvous and connectivity maintenance
   - Maintaining connectivity
   - Circumcenter algorithms
   - Correctness analysis via nondeterministic systems
3. Deployment
   - Expected-value deployment
   - Geometric-center laws
   - Disk-covering and sphere-packing deployment
4. Synchronized boundary patrolling
   - Balanced synchronization
   - Unbalanced synchronization
5. Conclusions

Rendezvous objective

Objective:
achieve multi-robot rendezvous; i.e. arrive at the same location of space, while maintaining connectivity

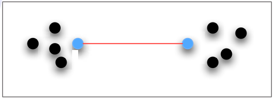


r-disk connectivity



visibility connectivity

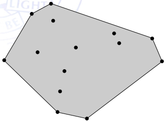
We have to be careful...



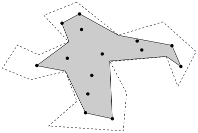
Blindly "getting closer" to neighboring agents might break overall connectivity

The rendezvous task via aggregate objective functions

Coordination task formulated as function minimization



Diameter convex hull



Perimeter relative convex hull

The rendezvous task formally

Let $\mathcal{S} = (\{1, \dots, n\}, \mathcal{R}, E_{\text{cmm}})$ be a uniform robotic network
The **(exact) rendezvous task** $\mathcal{T}_{\text{rendezvous}} : X^n \to \{\texttt{true}, \texttt{false}\}$ for $\mathcal{S}$ is

$$\mathcal{T}_{\text{rendezvous}}(x^{[1]}, \dots, x^{[n]}) = \begin{cases} \texttt{true}, & \text{if } x^{[i]} = x^{[j]}, \text{ for all } (i,j) \in E_{\text{cmm}}(x^{[1]}, \dots, x^{[n]}), \\ \texttt{false}, & \text{otherwise} \end{cases}$$

For $\epsilon \in \mathbb{R}_{>0}$, the **$\epsilon$-rendezvous task** $\mathcal{T}_{\epsilon\text{-rendezvous}} : (\mathbb{R}^d)^n \to \{\texttt{true}, \texttt{false}\}$ is

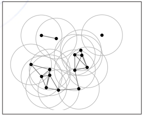
$$\mathcal{T}_{\epsilon\text{-rendezvous}}(P) = \texttt{true}$$
$$\iff \|p^{[i]} - \mathsf{avrg}\left(\left\{p^{[j]} \mid (i,j) \in E_{\text{cmm}}(P)\right\}\right)\|_2 < \epsilon, \quad i \in \{1, \dots, n\}$$

Outline

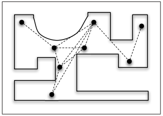
Constraint sets for connectivity

Design constraint sets with key properties

- Constraints are flexible enough so that network does not get stuck
- Constraints change continuously with agents' position



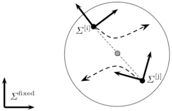
r-disk connectivity



visibility connectivity

Enforcing range-limited links – pairwise

Pairwise connectivity maintenance problem:
Given two neighbors in $\mathcal{G}_{\text{disk}}(r)$, find a rich set of control inputs for both agents with the property that, after moving, both agents are again within distance r



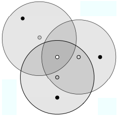
If $\|p^{[i]}(\ell) - p^{[j]}(\ell)\| \leq r$, and remain in ball of radius $r/2$ (connectivity set), then $\|p^{[i]}(\ell+1) - p^{[j]}(\ell+1)\| \leq r$

Enforcing range-limited links – w/ all neighbors

Definition (Connectivity constraint set)

Consider a group of agents at positions $P = \{p^{[1]}, \dots, p^{[n]}\} \subset \mathbb{R}^d$. The *connectivity constraint set* of agent i with respect to P is

$$\mathcal{X}_{\text{disk}}(p^{[i]}, P) = \bigcap \{\mathcal{X}_{\text{disk}}(p^{[i]}, q) \mid q \in P \setminus \{p^{[i]}\} \text{ s.t. } \|q - p^{[i]}\|_2 \leq r\}$$



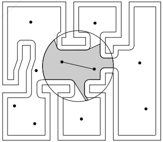
Same procedure over sparser graphs $\implies$ fewer constraints:

- select a graph that has same connected components
- select a graph whose edges can be computed in a distributed way

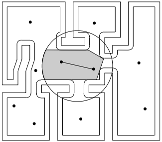
Enforcing range-limited line-of-sight links – pairwise

Given nonconvex $Q \subset \mathbb{R}^2$, contraction is $Q_\delta = \{q \in Q \mid \mathsf{dist}(q, \partial Q) \geq \delta\}$

Pairwise connectivity maintenance problem:
Given two neighbors in $\mathcal{G}_{\text{vis-disk},Q_\delta}$, find a rich set of control inputs for both agents with the property that, after moving, both agents are again within distance r and visible to each other in Q_δ



visibility region of agent i



visibility pairwise constraint set

Enforcing range-limited line-of-sight links – w/ all neighbors

Definition (Line-of-sight connectivity constraint set)

Consider a group of agents $P = \{p^{[1]}, \dots, p^{[n]}\}$ in nonconvex Q_δ. The **line-of-sight connectivity constraint sets** of agent i with respect to P is

$$\mathcal{X}_{\text{vis-disk}}(p^{[i]}, P; Q_\delta) = \bigcap \{\mathcal{X}_{\text{vis-disk}}(p^{[i]}, q; Q_\delta) \mid q \in P \setminus \{p^{[i]}\}\}$$

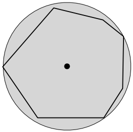
Fewer constraints can be generated via sparser graphs with the same connected components and spatially distributed

Outline

Circumcenter control and communication law

circumcenter $\mathsf{CC}(W)$ of bounded set W is center of closed ball of minimum radius containing W

Circumradius $\mathsf{CR}(W)$ is radius of this ball

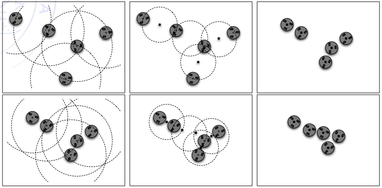


[Informal description:]

> *At each communication round each agent:*
> *(i) transmits its position and receives its neighbors' positions*
> *(ii) computes circumcenter of point set comprised of its neighbors and of itself*
> *(iii) moves toward this circumcenter point while remaining inside constraint set*

Circumcenter control and communication law

Illustration of the algorithm execution



Circumcenter control and communication law

Formal algorithm description

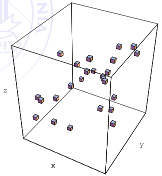
`Robotic Network:` $\mathcal{S}_{\rm disk}$ with a discrete-time motion model,
with absolute sensing of own position, and
with communication range r, in $\mathbb{R}^d$

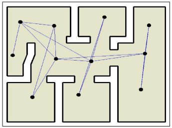
`Distributed Algorithm:` circumcenter
`Alphabet:` $L = \mathbb{R}^d \cup \{\texttt{null}\}$

`function` $\mathrm{msg}(p, i)$
return p

`function` $\mathrm{ctrl}(p, y)$
$p_{\rm goal} := \mathsf{CC}(\{p\} \cup \{p_{\rm rcvd} \mid \text{for all non-null } p_{\rm rcvd} \in y\})$
$\mathcal{X} := \mathcal{X}_{\rm disk}(p, \{p_{\rm rcvd} \mid \text{for all non-null } p_{\rm rcvd} \in y\})$
return $\mathsf{fti}(p, p_{\rm goal}, \mathcal{X}) - p$

Simulations





Outline

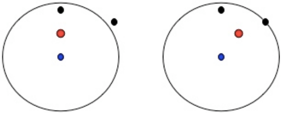
Some bad news...

Circumcenter algorithms are nonlinear discrete-time dynamical systems

$$x_{\ell+1} = f(x_\ell)$$

To analyze convergence, we need at least f continuous – to use classic Lyapunov/LaSalle results

But circumcenter algorithms are discontinuous because of changes in interaction topology



Alternative idea

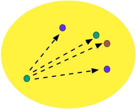
Fixed undirected graph G, define **fixed-topology circumcenter algorithm**

$$f_G : (\mathbb{R}^d)^n \to (\mathbb{R}^d)^n, \quad f_{G,i}(p_1,\dots,p_n) = \mathsf{fti}(p, p_{\text{goal}}, \mathcal{X}) - p$$

Now, there are no topological changes in f_G, hence f_G is continuous

Define set-valued map $T_{\mathcal{CC}} : (\mathbb{R}^d)^n \to \mathcal{P}((\mathbb{R}^d)^n)$

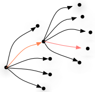
$$T_{\mathcal{CC}}(p_1,\dots,p_n) = \{f_G(p_1,\dots,p_n) \mid G \text{ connected}\}$$



Non-deterministic dynamical systems

Given $T : X \to \mathcal{P}(X)$, a **trajectory** of T is sequence $\{x_m\}_{m \in \mathbb{N}_0} \subset X$ such that

$$x_{m+1} \in T(x_m)\,, \quad m \in \mathbb{N}_0$$



T is **closed** at x if $x_m \to x$, $y_m \to y$ with $y_m \in T(x_m)$ imply $y \in T(x)$
Every continuous map $T \colon \mathbb{R}^d \to \mathbb{R}^d$ is closed on $\mathbb{R}^d$

A set C is

- **weakly positively invariant** if, for any $p_0 \in C$, there exists $p \in T(p_0)$ such that $p \in C$
- **strongly positively invariant** if, for any $p_0 \in C$, all $p \in T(p_0)$ verifies $p \in C$

A point p_0 is a *fixed point of* T if $p_0 \in T(p_0)$

LaSalle Invariance Principle – set-valued maps

$V \colon X \to \mathbb{R}$ is non-increasing along T on $S \subset X$ if

$$V(x') \leq V(x) \text{ for all } x' \in T(x) \text{ and all } x \in S$$

Theorem (LaSalle Invariance Principle)

For S compact and strongly invariant with V continuous and non-increasing along closed T on S

Any trajectory starting in S converges to largest weakly invariant set contained in $\{x \in S \mid \exists x' \in T(x) \text{ with } V(x') = V(x)\}$

Correctness

$T_{\mathcal{CC}}$ is closed and diameter is non-increasing

Recall set-valued map $T_{\mathcal{CC}} : (\mathbb{R}^d)^n \to \mathcal{P}((\mathbb{R}^d)^n)$

$$T_{\mathcal{CC}}(p_1, \dots, p_n) = \{f_{\mathcal{G}}(p_1, \dots, p_n) \mid \mathcal{G} \text{ connected}\}$$

$T_{\mathcal{CC}}$ is **closed**: finite combination of individual continuous maps
Define

$$V_{\text{diam}}(P) = \mathsf{diam}(\mathsf{co}(P)) = \max\{\|p_i - p_j\| \mid i, j \in \{1, \dots, n\}\}$$
$$\mathsf{diag}((\mathbb{R}^d)^n) = \{(p, \dots, p) \in (\mathbb{R}^d)^n \mid p \in \mathbb{R}^d\}$$

Lemma

The function $V_{\text{diam}} = \mathsf{diam} \circ \mathsf{co} \colon (\mathbb{R}^d)^n \to \overline{\mathbb{R}}_+$ verifies:

1. *V_{diam} is continuous and invariant under permutations;*
2. *$V_{\text{diam}}(P) = 0$ if and only if $P \in \mathsf{diag}((\mathbb{R}^d)^n)$;*
3. *V_{diam} is non-increasing along $T_{\mathcal{CC}}$*

Correctness via LaSalle Invariance Principle

To recap

1. $T_{\mathcal{CC}}$ is closed
2. $V = \mathsf{diam}$ is non-increasing along $T_{\mathcal{CC}}$
3. Evolution starting from P_0 is contained in $\mathsf{co}(P_0)$ (compact and strongly invariant)

Application of **LaSalle Invariance Principle:** trajectories starting at P_0 converge to M, largest weakly positively invariant set contained in

$$\{P \in \mathsf{co}(P_0) \mid \exists P' \in T_{\mathcal{CC}}(P) \text{ such that } \mathsf{diam}(P') = \mathsf{diam}(P)\}$$

Have to **identify** M! In fact, $M = \mathsf{diag}((\mathbb{R}^d)^n) \cap \mathsf{co}(P_0)$
Convergence to a point can be concluded with a little bit of extra work

Correctness

Theorem (Correctness of the circumcenter laws)

For $d \in \mathbb{N}$, $r \in \mathbb{R}_{>0}$ and $\epsilon \in \mathbb{R}_{>0}$, the following statements hold:

1. *on $\mathcal{S}_{\text{disk}}$, the law $\mathcal{CC}_{\text{circumcenter}}$ (with control magnitude bounds and relaxed $\mathcal{G}$-connectivity constraints) achieves $\mathcal{T}_{\text{rendezvous}}$;*
2. *on $\mathcal{S}_{\text{LD}}$, the law $\mathcal{CC}_{\text{circumcenter}}$ achieves $\mathcal{T}_{\epsilon\text{-rendezvous}}$*

Furthermore,

1. *if any two agents belong to the same connected component at $\ell \in \mathbb{N}_0$, then they continue to belong to the same connected component subsequently; and*
2. *for each evolution, there exists $P^* = (p_1^*, \dots, p_n^*) \in (\mathbb{R}^d)^n$ such that:*
   1. *the evolution asymptotically approaches P^*, and*
   2. *for each $i, j \in \{1, \dots, n\}$, either $p_i^* = p_j^*$, or $\|p_i^* - p_j^*\|_2 > r$*

Similar result for visibility networks in non-convex environments

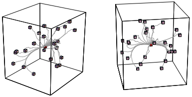
Correctness – Time complexity

Theorem (Time complexity of circumcenter laws)

For $r \in \mathbb{R}_{>0}$ and $\epsilon \in]0, 1[$, the following statements hold:

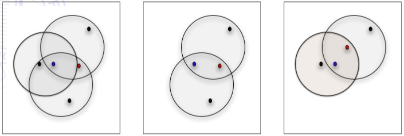
1. *on the network $\mathcal{S}_{\text{disk}}$, evolving on the real line $\mathbb{R}$ (i.e., with $d = 1$), $\mathsf{TC}(\mathcal{T}_{\text{rendezvous}}, \mathcal{CC}_{\text{circumcenter}}) \in \Theta(n)$;*
2. *on the network $\mathcal{S}_{\text{LD}}$, evolving on the real line $\mathbb{R}$ (i.e., with $d = 1$), $\mathsf{TC}(\mathcal{T}_{(r\epsilon)\text{-rendezvous}}, \mathcal{CC}_{\text{circumcenter}}) \in \Theta(n^2 \log(n\epsilon^{-1}))$; and*

Similar results for visibility networks



Robustness of circumcenter algorithms

Push whole idea further!, e.g., for robustness against link failures



topology G_1 topology G_2 topology G_3

Look at **evolution under link failures** as outcome of nondeterministic evolution under multiple interaction topologies

$$P \longrightarrow \{\text{evolution under } G_1, \text{evolution under } G_2, \text{evolution under } G_3\}$$

Rendezvous

Corollary (Circumcenter algorithm over $\mathcal{G}_{\text{disk}}(r)$ on $\mathbb{R}^d$)

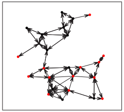
For $\{P_m\}_{m \in \mathbb{N}_0}$ synchronous execution with link failures such that union of any $\ell \in \mathbb{N}$ consecutive graphs in execution has globally reachable node

Then, there exists $(p^*, \dots, p^*) \in \mathsf{diag}((\mathbb{R}^d)^n)$ such that

$$P_m \to (p^*, \dots, p^*) \quad \text{as} \quad m \to +\infty$$

Proof uses

$T_{\mathcal{CC},\ell}(P) = \{f_{\mathcal{G}_\ell} \circ \cdots \circ f_{\mathcal{G}_1}(P) \mid$
$\cup_{s=1}^{\ell} \mathcal{G}_i$ has globally reachable node$\}$



Outline

Deployment

Objective: optimal task allocation and space partitioning
optimal placement and tuning of sensors

What notion of optimality? What algorithm design?

- **top-down approach:** define aggregate function measuring "goodness" of deployment, then synthesize algorithm that optimizes function
- **bottom-up approach:** synthesize "reasonable" interaction law among agents, then analyze network behavior

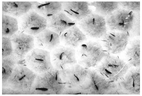
Coverage optimization

DESIGN of performance metrics

1. how to cover a region with n minimum-radius overlapping disks?
2. how to design a minimum-distortion (fixed-rate) vector quantizer? (Lloyd '57)
3. where to place mailboxes in a city / cache servers on the internet?

ANALYSIS of cooperative distributed behaviors

4. how do animals share territory? what if every fish in a swarm goes toward center of own dominance region?



Barlow, Hexagonal territories, *Animal Behavior*, 1974

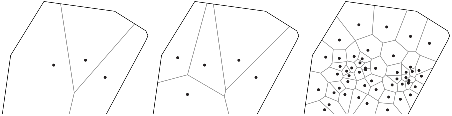
5. what if each vehicle goes to center of mass of own Voronoi cell?
6. what if each vehicle moves away from closest vehicle?

Voronoi partitions

Let $(p_1, \dots, p_n) \in Q^n$ denote the positions of n points

The **Voronoi partition** $\mathcal{V}(P) = \{V_1, \dots, V_n\}$ generated by $(p_1, \dots, p_n)$

$$\begin{aligned} V_i &= \{q \in Q \mid \|q - p_i\| \le \|q - p_j\|, \forall j \neq i\} \\ &= Q \cap_j \mathcal{HP}(p_i, p_j) \qquad \text{where } \mathcal{HP}(p_i, p_j) \text{ is half plane } (p_i, p_j) \end{aligned}$$

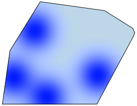


3 generators 5 generators 50 generators

Expected-value multicenter function

Objective: Given sensors/nodes/robots/sites $(p_1, \dots, p_n)$ moving in environment Q achieve **optimal coverage**

- $\phi \colon \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ density
- $f \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ non-increasing and piecewise continuously differentiable, possibly with finite jump discontinuities



$$\text{maximize} \quad \mathcal{H}_{\text{exp}}(p_1, \dots, p_n) = E_\phi \left[\max_{i \in \{1,\dots,n\}} f(\|q - p_i\|) \right]$$

Variety of scenarios

Alternative expression in terms of Voronoi partition,

$$\mathcal{H}_{\text{exp}}(p_1, \dots, p_n) = \sum_{i=1}^n \int_{V_i(P)} f(\|q - p_i\|_2) \phi(q) dq$$

Distortion problem: $f(x) = -x^2$ gives rise to ($\mathsf{J}_\phi(W,p)$ is moment of inertia)

$$\mathcal{H}_{\text{dist}}(p_1, \dots, p_n) = -\sum_{i=1}^n \mathsf{J}_\phi(V_i(P), p_i)$$

Area problem: $f(x) = 1_{[0,a]}(x)$, $a \in \mathbb{R}_{>0}$ gives rise to

$$\mathcal{H}_{\text{area},a}(p_1, \dots, p_n) = \sum_{i=1}^n \mathsf{area}_\phi(V_i(P) \cap \overline{B}(p_i, a)) = \mathsf{area}_\phi(\cup_{i=1}^n \overline{B}(p_i, a))$$

$\mathcal{H}_{\text{exp}}$-optimality of the Voronoi partition

$\mathcal{H}_{\text{exp}}$ as a function of agent positions and partition,

$$\mathcal{H}_{\text{exp}}(p_1, \dots, p_n, W_1, \dots, W_n) = \sum_{i=1}^n \int_{W_i} f(\|q - p_i\|_2) \phi(q) dq$$

Proposition (For fixed positions, Voronoi is optimal)

Let $P = \{p_1, \dots, p_n\} \in \mathbb{F}(S)$. For any performance function f and for any partition $\{W_1, \dots, W_n\} \subset \mathcal{P}(S)$ of S,

$$\mathcal{H}_{\text{exp}}(p_1, \dots, p_n, V_1(P), \dots, V_n(P)) \geq \mathcal{H}_{\text{exp}}(p_1, \dots, p_n, W_1, \dots, W_n),$$

and the inequality is strict if any set in $\{W_1, \dots, W_n\}$ differs from the corresponding set in $\{V_1(P), \dots, V_n(P)\}$ by a set of positive measure

Distortion problem

$f(x) = -x^2$

Using parallel axis theorem,

$$\mathcal{H}_{\text{dist}}(p_1, \dots, p_n, W_1, \dots, W_n) = -\sum_{i=1}^n \mathsf{J}_\phi(W_i, p_i)$$
$$= -\sum_{i=1}^n \mathsf{J}_\phi(W_i, \mathsf{CM}_\phi(W_i)) - \sum_{i=1}^n \mathsf{area}_\phi(W_i) \|p_i - \mathsf{CM}_\phi(W_i)\|_2^2$$

Proposition

Let $\{W_1, \dots, W_n\} \subset \mathcal{P}(S)$ be a partition of S. Then,

$$\mathcal{H}_{\text{dist}}(\mathsf{CM}_\phi(W_1), \dots, \mathsf{CM}_\phi(W_n), W_1, \dots, W_n) \geq \mathcal{H}_{\text{dist}}(p_1, \dots, p_n, W_1, \dots, W_n),$$

and the inequality is strict if there exists $i \in \{1, \dots, n\}$ for which W_i has non-vanishing area and $p_i \neq \mathsf{CM}_\phi(W_i)$

Gradient of $\mathcal{H}_{\rm exp}$ is distributed

For f smooth

$$\frac{\partial \mathcal{H}_{\rm exp}}{\partial p_i}(P) = \int_{V_i(P)} \frac{\partial}{\partial p_i} f\left(\|q-p_i\|\right)\phi(q)dq$$
$$+ \int_{\partial V_i(P)} f\left(\|q-p_i\|\right)\langle n_i(q), \frac{\partial q}{\partial p_i}\rangle \phi(q)dq$$
$$+ \underbrace{\sum_{j \text{ neigh } i} \int_{V_j(P)\cap V_i(P)} f\left(\|q-p_j\|\right)\langle n_{ji}(q), \frac{\partial q}{\partial p_i}\rangle \phi(q)dq}_{\text{contrib from neighbors}}$$

Gradient of $\mathcal{H}_{\rm exp}$ is distributed

For f smooth

$$\frac{\partial \mathcal{H}_{\rm exp}}{\partial p_i}(P) = \int_{V_i(P)} \frac{\partial}{\partial p_i} f\left(\|q-p_i\|\right)\phi(q)dq$$
$$+ \int_{\partial V_i(P)} f\left(\|q-p_i\|\right)\langle n_i(q), \frac{\partial q}{\partial p_i}\rangle \phi(q)dq$$
$$- \int_{\partial V_i(P)} f\left(\|q-p_i\|\right)\langle n_i(q), \frac{\partial q}{\partial p_i}\rangle \phi(q)dq$$

Therefore,

$$\frac{\partial \mathcal{H}_{\rm exp}}{\partial p_i}(P) = \int_{V_i(P)} \frac{\partial}{\partial p_i} f\left(\|q-p_i\|\right)\phi(q)dq$$

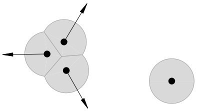
Particular gradients

Distortion problem: continuous performance,

$$\frac{\partial \mathcal{H}_{\rm dist}}{\partial p_i}(P) = 2\,\mathsf{area}_\phi(V_i(P))(\mathsf{CM}_\phi(V_i(P)) - p_i)$$

Area problem: performance has single discontinuity,

$$\frac{\partial \mathcal{H}_{{\rm area},a}}{\partial p_i}(P) = \int_{V_i(P)\,\cap\,\partial\overline{B}(p_i,a)} \mathsf{n}_{{\rm out},\overline{B}(p_i,a)}(q)\phi(q)dq$$



Smoothness properties of $\mathcal{H}_{\rm exp}$

$\mathsf{Dscn}(f)$ (finite) discontinuities of f
f_- and f_+, limiting values from the left and from the right

Theorem

Expected-value multicenter function $\mathcal{H}_{\rm exp}\colon S^n \to \mathbb{R}$ is

1. *globally Lipschitz on S^n; and*
2. *continuously differentiable on $S^n \setminus \mathcal{S}_{\rm coinc}$, where*

$$\frac{\partial \mathcal{H}_{\rm exp}}{\partial p_i}(P) = \int_{V_i(P)} \frac{\partial}{\partial p_i} f(\|q-p_i\|_2)\phi(q)dq$$
$$+ \sum_{a\in \mathsf{Dscn}(f)} (f_-(a) - f_+(a)) \int_{V_i(P)\,\cap\,\partial\overline{B}(p_i,a)} \mathsf{n}_{{\rm out},\overline{B}(p_i,a)}(q)\phi(q)dq$$
$$= \textit{integral over } V_i + \textit{integral along arcs in } V_i$$

Therefore, the gradient of $\mathcal{H}_{\rm exp}$ is spatially distributed over $\mathcal{G}_{\rm D}$

Outline

Geometric-center laws

Uniform networks $\mathcal{S}_{\mathrm{D}}$ and $\mathcal{S}_{\mathrm{LD}}$ of locally-connected first-order agents in a polytope $Q \subset \mathbb{R}^d$ with the Delaunay and r-limited Delaunay graphs as communication graphs

All laws share similar structure

At each communication round each agent performs:

- *it transmits its position and receives its neighbors' positions;*
- *it computes a notion of geometric center of its own cell determined according to some notion of partition of the environment*

Between communication rounds, each robot moves toward this center

VRN-CNTRD ALGORITHM

Optimizes distortion $\mathcal{H}_{\mathrm{dist}}$

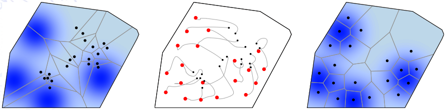
`Robotic Network:` $\mathcal{S}_{\mathrm{D}}$ in Q, with absolute sensing of own position

`Distributed Algorithm:` VRN-CNTRD
`Alphabet:` $L = \mathbb{R}^d \cup \{\texttt{null}\}$

`function` $\mathrm{msg}(p, i)$
1: **return** p

`function` $\mathrm{ctrl}(p, y)$
1: $V := Q \cap \left(\bigcap \{H_{p,p_{\mathrm{rcvd}}} \mid \text{for all non-null } p_{\mathrm{rcvd}} \in y\}\right)$
2: **return** $\mathsf{CM}_\phi(V) - p$

Simulation



initial configuration — gradient descent — final configuration

For $\epsilon \in \mathbb{R}_{>0}$, the *$\epsilon$-distortion deployment task*

$$\mathcal{T}_{\epsilon\text{-distor-dply}}(P) = \begin{cases} \texttt{true}, & \text{if } \left\| p^{[i]} - \mathsf{CM}_\phi(V^{[i]}(P)) \right\|_2 \le \epsilon,\ i \in \{1, \dots, n\}, \\ \texttt{false}, & \text{otherwise,} \end{cases}$$

Voronoi-centroid law on planar vehicles

`Robotic Network:` $\mathcal{S}_{\text{vehicles}}$ in Q with absolute sensing of own position
`Distributed Algorithm:` VRN-CNTRD-DYNMCS
`Alphabet:` $L = \mathbb{R}^2 \cup \{\texttt{null}\}$

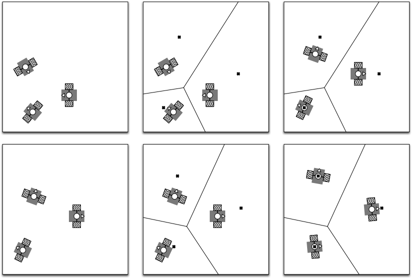
`function` $\text{msg}((p,\theta), i)$

1: **return** p

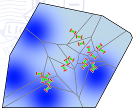
`function` $\text{ctrl}((p,\theta), (p_{\text{smpld}}, \theta_{\text{smpld}}), y)$

1: $V := Q \cap \left(\bigcap \{ H_{p_{\text{smpld}}, p_{\text{rcvd}}} \mid \text{for all non-null } p_{\text{rcvd}} \in y\} \right)$
2: $v := -k_{\text{prop}}(\cos\theta, \sin\theta) \cdot (p - \mathsf{CM}_\phi(V))$
3: $\omega := 2k_{\text{prop}} \arctan \dfrac{(-\sin\theta, \cos\theta)\cdot(p - \mathsf{CM}_\phi(V))}{(\cos\theta, \sin\theta)\cdot(p - \mathsf{CM}_\phi(V))}$
4: **return** (v, ω)

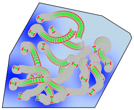
Algorithm illustration



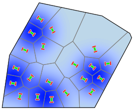
Simulation



initial configuration



gradient descent



final configuration

LMTD-VRN-NRML algorithm

Optimizes area $\mathcal{H}_{\text{area}, \frac{r}{2}}$

`Robotic Network:` $\mathcal{S}_{\text{LD}}$ in Q with absolute sensing of own position and with communication range r
`Distributed Algorithm:` LMTD-VRN-NRML
`Alphabet:` $L = \mathbb{R}^d \cup \{\texttt{null}\}$

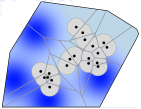
`function` $\text{msg}(p, i)$

1: **return** p

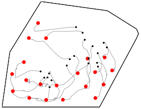
`function` $\text{ctrl}(p, y)$

1: $V := Q \cap \left(\bigcap \{ H_{p, p_{\text{rcvd}}} \mid \text{for all non-null } p_{\text{rcvd}} \in y\} \right)$
2: $v := \int_{V \cap \partial \overline{B}(p, \frac{r}{2})} \mathsf{n}_{\text{out}, \overline{B}(p, \frac{r}{2})}(q) \phi(q) dq$
3: $\lambda_* := \max \left\{ \lambda \mid \delta \mapsto \int_{V \cap \overline{B}(p + \delta v, \frac{r}{2})} \phi(q) dq \text{ is strictly increasing on } [0, \lambda] \right\}$
4: **return** $\lambda_* v$

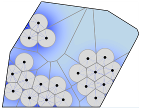
Simulation



initial configuration



gradient descent



final configuration

For $r, \epsilon \in \mathbb{R}_{>0}$,

$$\mathcal{T}_{\epsilon\text{-}r\text{-area-dply}}(P) = \begin{cases} \texttt{true}, & \text{if } \left\| \int_{V^{[i]}(P)\cap\partial\overline{B}(p^{[i]},\frac{r}{2})} \mathsf{n}_{\text{out},\overline{B}(p^{[i]},\frac{r}{2})}(q)\phi(q)dq \right\|_2 \le \epsilon,\ i \in \{1,\dots,n\}, \\ \texttt{false}, & \text{otherwise.} \end{cases}$$

Correctness of the geometric-center algorithms

Theorem

For $d \in \mathbb{N}$, $r \in \mathbb{R}_{>0}$ and $\epsilon \in \mathbb{R}_{>0}$, the following statements hold.

1. *on the network $\mathcal{S}_{\mathrm{D}}$, the law $\mathcal{CC}_{\text{VRN-CNTRD}}$ achieves the ϵ-distortion deployment task $\mathcal{T}_{\epsilon\text{-distor-dply}}$. Moreover, any execution monotonically optimizes $\mathcal{H}_{\text{dist}}$*
2. *on the network $\mathcal{S}_{\text{vehicles}}$, the law $\mathcal{CC}_{\text{VRN-CNTRD-DYNMCS}}$ achieves the ϵ-distortion deployment task $\mathcal{T}_{\epsilon\text{-distor-dply}}$. Moreover, any execution monotonically optimizes $\mathcal{H}_{\text{dist}}$*
3. *on the network $\mathcal{S}_{\mathrm{LD}}$, the law $\mathcal{CC}_{\text{LMTD-VRN-NRML}}$ achieves the ϵ-r-area deployment task $\mathcal{T}_{\epsilon\text{-}r\text{-area-dply}}$. Moreover, any execution monotonically optimizes $\mathcal{H}_{\text{area},\frac{r}{2}}$*

Time complexity of $\mathcal{CC}_{\text{LMTD-VRN-CNTRD}}$

Assume $\mathsf{diam}(Q)$ is independent of n, r and ϵ

Theorem (Time complexity of LMTD-VRN-CNTRD law)

Assume the robots evolve in a closed interval $Q \subset \mathbb{R}$, that is, $d = 1$, and assume that the density is uniform, that is, $\phi \equiv 1$. For $r \in \mathbb{R}_{>0}$ and $\epsilon \in \mathbb{R}_{>0}$, on the network $\mathcal{S}_{\mathrm{LD}}$

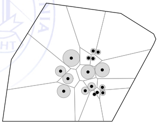
$$\mathsf{TC}(\mathcal{T}_{\epsilon\text{-}r\text{-distor-area-dply}}, \mathcal{CC}_{\text{LMTD-VRN-CNTRD}}) \in O(n^3 \log(n\epsilon^{-1}))$$

Open problem: characterize complexity of deployment algorithms in higher dimensions

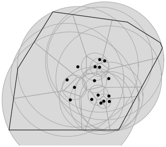
Outline

1. Models for multi-agent networks
2. Rendezvous and connectivity maintenance
   - Maintaining connectivity
   - Circumcenter algorithms
   - Correctness analysis via nondeterministic systems
3. Deployment
   - Expected-value deployment
   - Geometric-center laws
   - Disk-covering and sphere-packing deployment
4. Synchronized boundary patrolling
   - Balanced synchronization
   - Unbalanced synchronization
5. Conclusions

Deployment: basic behaviors



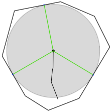
"move away from closest"

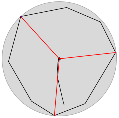


"move towards furthest"

Equilibria? Asymptotic behavior?
Optimizing network-wide function?

Deployment: 1-center optimization problems





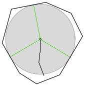
$\mathsf{sm}_Q(p) = \min\{\|p-q\| \mid q \in \partial Q\}$ Lipschitz $0 \in \partial\,\mathsf{sm}_Q(p) \Leftrightarrow p \in \mathsf{IC}(Q)$

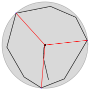
$\mathsf{lg}_Q(p) = \max\{\|p-q\| \mid q \in \partial Q\}$ Lipschitz $0 \in \partial\,\mathsf{lg}_Q(p) \Leftrightarrow p = \mathcal{CC}(Q)$

Locally Lipschitz function V are differentiable a.e.
Generalized gradient of V is

$$\partial V(x) = \text{convex closure}\{\lim_{i\to\infty} \nabla V(x_i) \mid x_i \to x\,,\ x_i \notin \Omega_V \cup S\}$$

Deployment: 1-center optimization problems





$+$ gradient flow of sm_Q $\quad \dot{p}_i = +\,\mathsf{Ln}[\partial\,\mathsf{sm}_Q](p)$ "move away from closest"

$-$ gradient flow of lg_Q $\quad \dot{p}_i = -\,\mathsf{Ln}[\partial\,\mathsf{lg}_Q](p)$ "move toward furthest"

For X essentially locally bounded, **Filippov solution** of $\dot{x} = X(x)$ is absolutely continuous function $t \in [t_0, t_1] \mapsto x(t)$ verifying

$$\dot{x} \in K[X](x) = \mathsf{co}\{\lim_{i\to\infty} X(x_i) \mid x_i \to x\,,\ x_i \notin S\}$$

For V locally Lipschitz, gradient flow is $\dot{x} = \mathsf{Ln}[\partial V](x)$
Ln = least norm operator

Nonsmooth LaSalle Invariance Principle

Evolution of V along Filippov solution $t \mapsto V(x(t))$ is differentiable a.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) \in \underbrace{\widetilde{\mathcal{L}}_X V(x(t)) = \{a \in \mathbb{R} \mid \exists v \in K[X](x) \text{ s.t. } \zeta \cdot v = a\,,\ \forall \zeta \in \partial V(x)\}}_{\text{set-valued Lie derivative}}$$

LaSalle Invariance Principle

For S compact and strongly invariant with $\max \widetilde{\mathcal{L}}_X V(x) \leq 0$
Any Filippov solution starting in S converges to largest weakly invariant set contained in $\overline{\left\{x \in S \mid 0 \in \widetilde{\mathcal{L}}_X V(x)\right\}}$

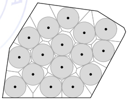
E.g., **nonsmooth gradient flow** $\dot{x} = -\,\mathsf{Ln}[\partial V](x)$ converges to critical set

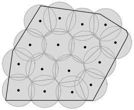
Deployment: multi-center optimization

sphere packing and disk covering

"move away from closest": $\dot{p}_i = +\,\mathsf{Ln}(\partial\,\mathsf{sm}_{V_i(P)})(p_i)$ — at fixed $V_i(P)$

"move towards furthest": $\dot{p}_i = -\,\mathsf{Ln}(\partial\,\mathsf{lg}_{V_i(P)})(p_i)$ — at fixed $V_i(P)$





Aggregate objective functions!

$$\mathcal{H}_{\mathrm{sp}}(P) = \min_i \; \mathsf{sm}_{V_i(P)}(p_i) = \min_{i \neq j} \left[\tfrac{1}{2}\|p_i - p_j\|, \mathsf{dist}(p_i, \partial Q)\right]$$

$$\mathcal{H}_{\mathrm{dc}}(P) = \max_i \; \mathsf{lg}_{V_i(P)}(p_i) = \max_{q \in Q} \left[\min_i \|q - p_i\|\right]$$

Deployment: multi-center optimization

Critical points of $\mathcal{H}_{\mathbf{sp}}$ and $\mathcal{H}_{\mathbf{dc}}$ (locally Lipschitz)

- If $0 \in \mathsf{int}\,\partial\mathcal{H}_{\mathrm{sp}}(P)$, then P is strict local maximum, all agents have same cost, and P is **incenter Voronoi configuration**
- If $0 \in \mathsf{int}\,\partial\mathcal{H}_{\mathrm{dc}}(P)$, then P is strict local minimum, all agents have same cost, and P is **circumcenter Voronoi configuration**

Aggregate functions **monotonically optimized** along evolution

$$\min \widetilde{\mathcal{L}}_{\mathsf{Ln}(\partial\,\mathsf{sm}_{\mathcal{V}(P)})}\mathcal{H}_{\mathbf{sp}}(P) \geq 0 \qquad \max \widetilde{\mathcal{L}}_{-\,\mathsf{Ln}(\partial\,\mathsf{lg}_{\mathcal{V}(P)})}\mathcal{H}_{\mathbf{dc}}(P) \leq 0$$

Asymptotic convergence via nonsmooth LaSalle principle

- Convergence to configurations where all agents whose local cost coincides with aggregate cost are centered
- Convergence to center Voronoi configurations still open

Voronoi-circumcenter algorithm

`Robotic Network:` $\mathcal{S}_{\mathrm{D}}$ in Q with absolute sensing of own position

`Distributed Algorithm:` VRN-CRCMCNTR

`Alphabet:` $L = \mathbb{R}^d \cup \{\mathtt{null}\}$

`function` msg(p, i)

1: **return** p

`function` ctrl(p, y)

1: $V := Q \cap \left(\bigcap\{H_{p,p_{\mathrm{rcvd}}} \mid \text{for all non-null } p_{\mathrm{rcvd}} \in y\}\right)$

2: **return** $\mathsf{CC}(V) - p$

Voronoi-incenter algorithm

`Robotic Network:` $\mathcal{S}_{\mathrm{D}}$ in Q with absolute sensing of own position

`Distributed Algorithm:` VRN-NCNTR

`Alphabet:` $L = \mathbb{R}^d \cup \{\mathtt{null}\}$

`function` msg(p, i)

1: **return** p

`function` ctrl(p, y)

1: $V := Q \cap \left(\bigcap\{H_{p,p_{\mathrm{rcvd}}} \mid \text{for all non-null } p_{\mathrm{rcvd}} \in y\}\right)$

2: **return** $x \in \mathsf{IC}(V) - p$

Correctness of the geometric-center algorithms

For $\epsilon \in \mathbb{R}_{>0}$, the *$\epsilon$-disk-covering deployment task*

$$\mathcal{T}_{\epsilon\text{-dc-dply}}(P) = \begin{cases} \texttt{true}, & \text{if } \|p^{[i]} - \mathsf{CC}(V^{[i]}(P))\|_2 \leq \epsilon,\ i \in \{1,\dots,n\}, \\ \texttt{false}, & \text{otherwise}, \end{cases}$$

For $\epsilon \in \mathbb{R}_{>0}$, the *$\epsilon$-sphere-packing deployment task*

$$\mathcal{T}_{\epsilon\text{-sp-dply}}(P) = \begin{cases} \texttt{true}, & \text{if } \mathsf{dist}_2(p^{[i]}, \mathsf{IC}(V^{[i]}(P))) \leq \epsilon,\ i \in \{1,\dots,n\}, \\ \texttt{false}, & \text{otherwise}, \end{cases}$$

Theorem

For $d \in \mathbb{N}$, $r \in \mathbb{R}_{>0}$ and $\epsilon \in \mathbb{R}_{>0}$, the following statements hold.

1. *on the network $\mathcal{S}_{\mathrm{D}}$, any execution of the law $\mathcal{CC}_{\text{VRN-CRCMCNTR}}$ monotonically optimizes the multicenter function $\mathcal{H}_{\mathrm{dc}}$;*
2. *on the network $\mathcal{S}_{\mathrm{D}}$, any execution of the law $\mathcal{CC}_{\text{VRN-NCNTR}}$ monotonically optimizes the multicenter function $\mathcal{H}_{\mathrm{sp}}$.*

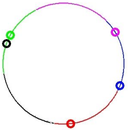
Outline

Synchronized boundary patrolling

Joint work with Sara Susca (Honeywell)
and Sonia Martínez (UCSD)

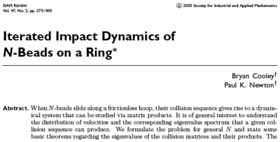
1. some UAVs move along boundary of sensitive territory
2. short-range communication and sensing
3. surveillance objective:
   - minimize service time for appearing events
   - communication network connectivity

Example motion:



Analogy with mechanics and dynamics

1. robots with "communication impacts" analogous to beads on a ring
2. classic subject in dynamical systems and geometric mechanics
   - billiards, iterated impact dynamics, gas theory of hard spheres
3. rich dynamics with even just 3 beads (distinct masses, elastic collisions)
   - dynamics akin billiard flow inside acute triangle
   - dense periodic and nonperiodic modes, chaotic collision sequences



SIAM REVIEW
Vol. 47, No. 2, pp. 273–300

© 2005 Society for Industrial and Applied Mathematics

Iterated Impact Dynamics of *N*-Beads on a Ring*

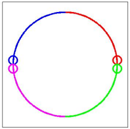
Bryan Cooley†
Paul K. Newton†

Abstract. When *N*-beads slide along a frictionless hoop, their collision sequence gives rise to a dynamical system that can be studied via matrix products. It is of general interest to understand the distribution of velocities and the corresponding eigenvalue spectrum that a given collision sequence can produce. We formulate the problem for general N and state some basic theorems regarding the eigenvalues of the collision matrices and their products. The

Boundary patrolling: synchronized bead oscillation

Desired synchronized behavior:

- starting from random initial positions and velocities
- every bead impacts its neighbor at the same point
- simultaneous impacts



Design specification for synchronization algorithm

Achieve: asymptotically stabilize synchronized motion
Subject to:

1. arbitrary initial positions, velocities and directions of motion
2. beads can measure traveled distance, however
   no absolute localization capability, no knowledge of circle length
3. no knowledge about n, adaptation to changing n (even and odd)
4. anynomous agents with memory and message sizes independent of n
5. smooth dependency upon effect of measurement and control noise

Fully-adaptive feedback synchronization

Slowdown-Impact-Speedup algorithm

Algorithm: (for presentation's sake, beads sense their position)

1st phase: compute **average speed** v and **desired sweeping arcs**

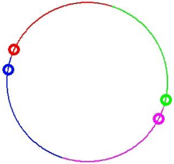
2nd phase for $f \in]\frac{1}{2}, 1[$, each bead:

- moves at nominal speed v if inside its desired sweeping arc
- slows down (fv) when moving away of its sweeping arc
  hesitate when early
- when impact, change direction
- speeds up when moving towards its desired sweeping arc

Simulations results: balanced synchronization

Balanced initial condition:

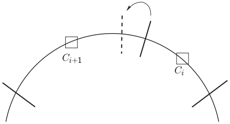
- n is even
- d_i is direction of motion
- $\sum_i^n d_i(0) = \sum_i^n d_i(t) = 0$
- $n/2$ move initially clockwise



First phase: average speed and sweeping arc

If an impact between bead i and $i+1$ occurs:

- beads average nominal speeds: $v_i^+ = v_{i+1}^+ = 0.5(v_i + v_{i+1})$
- beads change their direction of motion if $d_i = -d_{i+1}$ (**head-head type**)
- beads update their desired sweeping arc



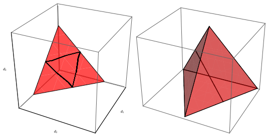
exponential average consensus

Challenges

1. how to prove balanced synchronization?
2. what happens for unbalanced initial conditions $\sum_i^n d_i(0) \neq 0$?
3. what happens for n is odd?
4. how to describe the system with a single variable?

Modeling detour

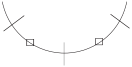
- configuration space
  1. order-preserving dynamics $\theta_i \in \mathrm{Arc}(\theta_{i-1}, \theta_{i+1})$ on $\mathbb{T}^n$
  2. $\Delta^n \times \{\mathrm{c}, \mathrm{cc}\}^n \times (\mathbb{R}_{>0})^n \times (\mathrm{arcs})^n \times \{\mathrm{away}, \mathrm{towards}\}^n$



- hybrid system with
  1. piecewise constant dynamics
  2. event-triggered jumps: impact, cross boundary

Passage and return times

- passage time: t_i^k = kth time when bead i passes by sweeping arc center



- return time: $\delta_i(t)$ = **duration between last two passage times**
- if impact between beads $(i, i+1)$ at time t, then

$$\begin{bmatrix} \delta_i \\ \delta_{i+1} \end{bmatrix}(t^+) = \underbrace{\begin{bmatrix} \frac{1-f}{1+f} & \frac{2f}{1+f} \\ \frac{2f}{1+f} & \frac{1-f}{1+f} \end{bmatrix}}_{\text{stochastic}} \begin{bmatrix} \delta_i \\ \delta_{i+1} \end{bmatrix}(t^-)$$

Averaging algorithms

Distributed averaging algorithm or **consensus algorithms**

$$x(\ell+1) = Ax(\ell)$$

with **(row) stochastic** matrix A: $\sum_{j=1}^{n} a_{ij} = 1$ and $a_{ij} \geq 0$

- let $G(A)$ be unweighted matrix associated to A
- a sequence of stochastic $\{A(\ell)\}_{\ell\in\mathbb{N}}$ is **non-degenerate** if $\exists\alpha > 0$ s.t. $a_{ii}(\ell) \geq \alpha$ and $a_{ij}(\ell) \in \{0\} \cup [\alpha, 1]$, for all $i \neq j$

Theorem (Convergence to average consensus)

Let $\{A(\ell)\}_{\ell\in\mathbb{N}}$ be a non-degenerate sequence of stochastic, symmetric matrices

1. *each evolution x converges to* $\text{average}(x(0))\mathbf{1}_n$
2. *for all $\ell \in \mathbb{N}$, the graph $\bigcup_{\tau\geq\ell} G(A(\tau))$ is connected*

Convergence results: balanced synchronization

Balanced Synchronization Theorem: For balanced initial directions, assume

1. exact average speed and desired sweeping arcs
2. initial conditions lead to well-defined 1st passage times

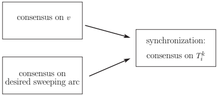
Then balanced synchronization is asymptotically stable

$$\lim_{t\to\infty} \delta(t) = \frac{2\pi}{Nv}\mathbf{1}_n, \qquad \lim_{k\to+\infty} \left\| T^k - \frac{\mathbf{1}_n \cdot T^k}{n}\mathbf{1}_n \right\| = 0$$

Conjectures arising from simulation results

Only assumption required is balanced initial conditions.

1. analysis of cascade consensus algorithms

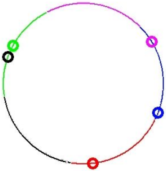


2. global attractivity of synchronous behavior

Simulations results: 1-unbalanced case

1-unbalanced initial condition:

- n is odd
- $\sum_i^n d_i(0) = \sum_i^n d_i(t) = \pm 1$



1-unbalanced synchronization

1. $f \in]\frac{1}{2}, \frac{n}{1+n}[$
2. 1-unbalanced sync: beads meet at arcs boundaries $\pm\frac{2\pi}{n^2}\frac{f}{1-f}$

1-unbalanced Synchronization Theorem: For $\sum_i^n d_i(0) = \pm 1$, assume

1. exact average speed and desired sweeping arcs
2. initial conditions lead to well-defined 1st passage times

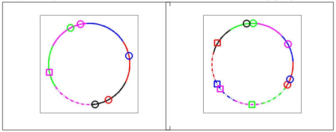
Then 1-unbalanced synchronization is asymptotically stable

$$\lim_{t\to\infty} \delta(t) = \frac{2\pi}{Nv}\mathbf{1}_n, \qquad \lim_{k\to+\infty} \left(T^{2k} - T^{2(k-1)}\right) = \frac{2}{v}\frac{2\pi}{n}\mathbf{1}_n$$

General unbalanced case

Conjecture global asy-synchronization in the balanced and unbalanced case

D-unbalanced period orbits Theorem:
Let $\sum_i^n d_i(0) = \pm D$. If there exists an orbit along which beads i and $i+1$ meet at boundary $\pm\frac{2\pi}{n^2}\frac{f}{1-f}$, then $f < \frac{n/|D|}{1+n/|D|}$.



Summary and conclusions

Examined various motion coordination tasks

1. **rendezvous:** circumcenter algorithms
2. **connectivity maintenance:** flexible constraint sets in convex/nonconvex scenarios
3. **deployment:** gradient algorithms based on geometric centers
4. **beads problem:** robotic patrolling via synchronization

Correctness and **(1-d) complexity analysis** of geometric-center control and communication laws via

1. Discrete- and continuous-time nondeterministic dynamical systems
2. Invariance principles, stability analysis
3. Geometric structures and geometric optimization

Plenty of open problems!

Motion coordination is emerging discipline

Literature is full of exciting problems, solutions, and tools we have not covered

Formation control, consensus, cohesiveness, flocking, collective synchronization, boundary estimation, cooperative control over constant graphs, quantization, asynchronism, delays, distributed estimation, spatial estimation, data fusion, target tracking, networks with minimal capabilities, target assignment, vehicle dynamics and energy-constrained motion, vehicle routing, dynamic servicing problems, load balancing, robotic implementations,...

Too long a list to fit it here!