Cooperative multi-agent systems

What kind of systems?
Groups of agents with control, sensing, communication and computing

Each individual
- **senses** its immediate environment
- **communicates** with others
- **processes** information gathered
- **takes local action** in response

Self-organized behaviors in biological groups

Decision making in animals

Able to
- deploy over a given region
- assume specified pattern
- rendezvous at a common point
- jointly initiate motion/change direction in a synchronized way

Species achieve synchronized behavior
- with limited sensing/communication between individuals
- without apparently following group leader

(Couzin et al, Nature 05; Conradt et al, Nature 03)
Engineered multi-agent systems

Embedded robotic systems and sensor networks for
- high-stress, rapid deployment — e.g., disaster recovery networks
- distributed environmental monitoring — e.g., portable chemical and biological sensor arrays detecting toxic pollutants
- autonomous sampling for biological applications — e.g., monitoring of species in risk, validation of climate and oceanographic models
- science imaging — e.g., multiscapcecraft distributed interferometers flying in formation to enable imaging at microarcsecond resolution

Research challenges

What useful engineering tasks can be performed with limited-sensing/communication agents?

Dynamics
simple interactions give rise to rich emerging behavior
Feedback
rather than open-loop computation for known/static setup
Information flow
who knows what, when, why, how, dynamically changing
Reliability/performance
robust, efficient, predictable behavior

How to coordinate individual agents into coherent whole?

Objective: systematic methodologies to design and analyze cooperative strategies to control multi-agent systems

Research program: what are we after?

Design of provably correct coordination algorithms for basic tasks

Formal model to rigorously formalize, analyze, and compare coordination algorithms

Mathematical tools to study convergence, stability, and robustness of coordination algorithms

Coordination tasks
exploration, map building, search and rescue, surveillance, odor localization, monitoring, distributed sensing

Technical approach

Optimization Methods
- resource allocation
- geometric optimization
- load balancing

Geometry & Analysis
- computational structures
- differential geometry
- nonsmooth analysis

Control & Robotics
- algorithm design
- cooperative control
- stability theory

Distributed Algorithms
- adhoc networks
- decentralized vs centralized
- emerging behaviors
Fig. 2.2 A top-view photograph, using a polarizing filter, of the territories of the male Tilapia mossambica. Photograph and caption reprinted from G. W. Barlow, Cichlasoma maculicauda. Photograph and caption reprinted from G. W. Barlow, Cichlasoma maculicauda. Photograph and caption reprinted from G. W. Barlow, Cichlasoma maculicauda.

Territorial males of this species create breeding pits, i.e., the locations at which they will spit sand. The desire to place the pit centers as far away as possible from their neighbors causes the fish to continuously adjust the position of the pit centers. This adjustment process is modeled as follows. Since all the fish are assumed to be of equal strength, i.e., they all presumably have the same ability to excavate breeding pits, results in the observation that they are very closely approximated by a Voronoi tessellation. The hexagonal territories are seen to be polygonal, and, in [27,59], it was shown that they are very closely approximated by a Voronoi tessellation.

A behavioral model for how the fish establish their territories was given in [22,60]. When the fish enter a region, they randomly select the centers of their territories. As an example of synchronous settling for which the territories can be visualized, an observation is reproduced. The territories are seen to be polygonal, and, in [27,59], it was shown that they are very closely approximated by a Voronoi tessellation.

Robustness against link failures, agents’ arrivals and departures, delays, asynchronism

Control/communication tradeoffs

Feasible operations and their cost

Meaningful + tractable model

Design coordination algorithms that achieve these tasks and analyze their correctness and time complexity

Expand set of math tools: invariance principles for non-deterministic systems, geometric optimization, non-smooth stability analysis

Robustness against link failures, agents’ arrivals and departures, delays, asynchronism

Introduction to distributed algorithms

Graph theory, synchronous networks, and complexity

Geometric models and geometric optimization problems

Model for robotic, relative sensing networks, and complexity

Algorithms for rendezvous, deployment, boundary estimation

Status: Freely downloadable at http://coordinationbook.info

with tutorial slides & software libraries. Shortly on sale by Princeton Univ Press
A uniform/anonymous robotic network $S$ is

- $I = \{1, \ldots, N\}$; set of unique identifiers (UIDs)
- $A = \{A[i]\}_{i \in I}$, with $A[i] = (X, U, f)$ is a set of physical agents
- interaction graph

Disk, visibility and Delauney graphs

Relevant graphs
- fixed, directed, balanced
- switching
- geometric or state-dependent
- random, random geometric

Message model
- message
- packet/bits
- absolute or relative positions
- packet losses

Prototypical examples

Locally-connected first-order robots in $\mathbb{R}^d$ $S_{\text{disk}}$

- $n$ points $x[1], \ldots, x[n]$ in $\mathbb{R}^d$, $d \geq 1$
- obeying $\dot{x}[i](t) = u[i](t)$, with $u[i] \in [-u_{\text{max}}, u_{\text{max}}]$
- identical robots of the form
  $$(\mathbb{R}^d, [-u_{\text{max}}, u_{\text{max}}]^d, \mathbb{R}^d, (0, e_1, \ldots, e_d))$$
- each robot communicates to other robots within $r$

Variations
- $S_D$: same dynamics, but Delaunay graph
- $S_{LD}$: same dynamics, but $r$-limited Delaunay graph
- $S_{\text{vehicles}}$: same graph, but nonholonomic dynamics

Synchronous control and communication

- communication schedule $T = \{t_\ell\}_{\ell \in \mathbb{N}_0} \subset \mathbb{R}_{\geq 0}$
- communication alphabet $L$ including the null message $W$
- set of values for logic variables $W$
- message-generation function $\text{msg}: T \times X \times W \times I \rightarrow L$
- state-transition functions $\text{stf}: T \times X \times L^N \rightarrow W$
- control function $\text{ctrl}: \mathbb{R}_{\geq 0} \times X \times W \times L^N \rightarrow U$
Task and complexity

- **Coordination task** is \((W, T)\) where \(T: X^N \times W^N \rightarrow \{\text{true}, \text{false}\}\)
  - Logic-based: achieve consensus, synchronize, form a team
  - Motion: deploy, gather, flock, reach pattern
  - Sensor-based: search, estimate, identify, track, map

- For \({\mathcal{S}, T, \mathcal{C}\mathcal{C}}\), define costs/complexity:
  - control effort, communication packets, computational cost
- **Time complexity to achieve \(T\) with \(\mathcal{C}\mathcal{C}\)**
  \[
  TC(T, \mathcal{C}\mathcal{C}, x_0, w_0) = \inf \{ \ell | T(x(t_k), w(t_k)) = \text{true}, \text{ for all } k \geq \ell \}
  \]
  \[
  TC(T, \mathcal{C}\mathcal{C}) = \sup \{ TC(T, \mathcal{C}\mathcal{C}, x_0, w_0) | (x_0, w_0) \in X^N \times W^N \}
  \]
  \[
  TC(T) = \inf \{ TC(T, \mathcal{C}\mathcal{C}) | \mathcal{C}\mathcal{C} \text{ achieves } T \}
  \]

Outline

1. Models for multi-agent networks
2. Rendezvous and connectivity maintenance
   - Maintaining connectivity
   - Circumcenter algorithms
   - Correctness analysis via nondeterministic systems
3. Deployment
   - Expected-value deployment
   - Geometric-center laws
   - Disk-covering and sphere-packing deployment
4. Synchronized boundary patrolling
   - Balanced synchronization
   - Unbalanced synchronization
5. Conclusions

Rendezvous objective

**Objective:**
achieve multi-robot rendezvous; i.e. arrive at the same location of space, while maintaining connectivity

Blindly “getting closer” to neighboring agents might break overall connectivity
The rendezvous task via aggregate objective functions

Coordination task formulated as function minimization

- Diameter convex hull
- Perimeter relative convex hull

Bullo & Cortés (UCSB/UCSD)
Distributed Coordination Algorithms
May 17, 2009 21 / 97

The rendezvous task formally

Let $S = (\{1, \ldots, n\}, R, E_{cmm})$ be a uniform robotic network.

The (exact) rendezvous task $T_{\text{rendezvous}}: \mathbb{R}^n \rightarrow \{\text{true}, \text{false}\}$ for $S$ is

$$T_{\text{rendezvous}}(x^[1], \ldots, x^[n]) = \begin{cases} \text{true}, & \text{if } x^[i] = x^[j], \text{ for all } (i, j) \in E_{cmm}(x^[1], \ldots, x^[n]), \\ \text{false}, & \text{otherwise} \end{cases}$$

For $\epsilon \in \mathbb{R}_{>0}$, the $\epsilon$-rendezvous task $T_{\epsilon, \text{rendezvous}}: (\mathbb{R}^d)^n \rightarrow \{\text{true}, \text{false}\}$ is

$$T_{\epsilon, \text{rendezvous}}(P) = \text{true} \iff \|p^[i] - \text{avg}(\{p^[j] | (i, j) \in E_{cmm}(P)\})\|_2 < \epsilon, \quad i \in \{1, \ldots, n\}$$

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Constraint sets for connectivity

Design constraint sets with key properties
- Constraints are flexible enough so that network does not get stuck
- Constraints change continuously with agents’ position

$r$-disk connectivity

visibility connectivity
Enforcing range-limited links – pairwise

**Pairwise connectivity maintenance problem:**
Given two neighbors in $G_{\text{disk}}(r)$, find a rich set of control inputs for both agents with the property that, after moving, both agents are again within distance $r$.

If $\|p^{[i]}(\ell) - p^{[j]}(\ell)\| \leq r$, and remain in ball of radius $r/2$ (connectivity set), then $\|p^{[i]}(\ell + 1) - p^{[j]}(\ell + 1)\| \leq r$.

Enforcing range-limited line-of-sight links – pairwise

Given nonconvex $Q \subset \mathbb{R}^2$, contraction is $Q_\delta = \{q \in Q \mid \text{dist}(q, \partial Q) \geq \delta\}$.

**Pairwise connectivity maintenance problem:**
Given two neighbors in $G_{\text{vis-disk},Q_\delta}$, find a rich set of control inputs for both agents with the property that, after moving, both agents are again within distance $r$ and visible to each other in $Q_\delta$.

Enforcing range-limited line-of-sight links – w/ all neighbors

**Definition (Line-of-sight connectivity constraint set)**
Consider a group of agents at positions $P = \{p^{[1]}, \ldots, p^{[n]}\} \subset \mathbb{R}^d$. The line-of-sight connectivity constraint sets of agent $i$ with respect to $P$ is

$$X_{\text{vis-disk}}(p^{[i]}, P) = \bigcap \{X_{\text{vis-disk}}(p^{[j]}, q) \mid q \in P \setminus \{p^{[i]}\} \text{ s.t. } \|q - p^{[i]}\|_2 \leq r\}$$

Fewer constraints can be generated via sparser graphs with the same connected components and spatially distributed.
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Circumcenter control and communication law

circumcenter CC(W) of bounded set W is center of closed ball of minimum radius containing W

Circumradius CR(W) is radius of this ball

[Informal description:]
At each communication round each agent:
(i) transmits its position and receives its neighbors’ positions
(ii) computes circumcenter of point set comprised of its neighbors and of itself
(iii) moves toward this circumcenter point while remaining inside constraint set

Circumcenter control and communication law
Illustration of the algorithm execution

Formal algorithm description

Robotic Network: S_{disk} with a discrete-time motion model, with absolute sensing of own position, and with communication range r, in \mathbb{R}^d

Distributed Algorithm: circumcenter

Alphabet: \mathcal{L} = \mathbb{R}^d \cup \{null\}

function msg(p, i)
1: return p

function ctrl(p, y)
1: \text{p}_{\text{goal}} := CC(\{p\} \cup \{p_{\text{rcvd}} | \text{for all non-null } p_{\text{rcvd}} \in y\})
2: \mathcal{X} := X_{\text{disk}}(p, \{p_{\text{rcvd}} | \text{for all non-null } p_{\text{rcvd}} \in y\})
3: \text{return } fti(p, \text{p}_{\text{goal}}, \mathcal{X}) - p
Simulations

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Some bad news...

Circumcenter algorithms are nonlinear discrete-time dynamical systems

\[ x_{t+1} = f(x_t) \]

To analyze convergence, we need at least \( f \) continuous – to use classic Lyapunov/LaSalle results

But circumcenter algorithms are discontinuous because of changes in interaction topology

Alternative idea

Fixed undirected graph \( G \), define fixed-topology circumcenter algorithm

\[ f_G : (\mathbb{R}^d)^n \rightarrow (\mathbb{R}^d)^n, \quad f_G,i(p_1, \ldots, p_n) = f_{ti}(p, p_{\text{goal}}, \mathcal{X}) - p \]

Now, there are no topological changes in \( f_G \), hence \( f_G \) is continuous

Define set-valued map \( T_{CC} : (\mathbb{R}^d)^n \rightarrow \mathcal{P}( (\mathbb{R}^d)^n ) \)

\[ T_{CC}(p_1, \ldots, p_n) = \{ f_G(p_1, \ldots, p_n) \mid G \text{ connected} \} \]
Given $T : X \to \mathcal{P}(X)$, a trajectory of $T$ is a sequence $\{x_m\}_{m \in \mathbb{N}_0} \subset X$ such that

$$x_{m+1} \in T(x_m), \quad m \in \mathbb{N}_0$$

$T$ is closed at $x$ if $x_m \to x$, $y_m \to y$ with $y_m \in T(x_m)$ imply $y \in T(x)$

Every continuous map $T : \mathbb{R}^d \to \mathbb{R}^d$ is closed on $\mathbb{R}^d$

A set $C$ is

- weakly positively invariant if, for any $p_0 \in C$, there exists $p \in T(p_0)$ such that $p \in C$

- strongly positively invariant if, for any $p_0 \in C$, all $p \in T(p_0)$ verifies $p \in C$

A point $p_0$ is a fixed point of $T$ if $p_0 \in T(p_0)$

**Correctness**

$T_{\text{CC}}$ is closed and diameter is non-increasing

Recall set-valued map $T_{\text{CC}} : (\mathbb{R}^d)^n \to \mathcal{P}((\mathbb{R}^d)^n)$

$$T_{\text{CC}}(p_1, \ldots, p_n) = \{ f_G(p_1, \ldots, p_n) \mid G \text{ connected} \}$$

$T_{\text{CC}}$ is closed: finite combination of individual continuous maps

Define

$$V_{\text{diam}}(P) = \text{diam}(\text{co}(P)) = \max \{ \| p_i - p_j \| \mid i, j \in \{1, \ldots, n\} \}$$

$$\text{diag}((\mathbb{R}^d)^n) = \{(p, \ldots, p) \in (\mathbb{R}^d)^n \mid p \in \mathbb{R}^d \}$$

**Lemma**

The function $V_{\text{diam}} = \text{diam} \circ \text{co} : (\mathbb{R}^d)^n \to \mathbb{R}_+$ verifies:

- $V_{\text{diam}}$ is continuous and invariant under permutations;
- $V_{\text{diam}}(P) = 0$ if and only if $P \in \text{diag}((\mathbb{R}^d)^n)$;
- $V_{\text{diam}}$ is non-increasing along $T_{\text{CC}}$
Theorem (Correctness of the circumcenter laws)

For \( d \in \mathbb{N}, r \in \mathbb{R}_{>0} \) and \( \epsilon \in \mathbb{R}_{>0} \), the following statements hold:

1. on \( S_{\text{disk}} \), the law \( CC_{\text{circumcenter}} \) (with control magnitude bounds and relaxed \( G \)-connectivity constraints) achieves \( T_{\text{rendezvous}} \);
2. on \( S_{\text{LD}} \), the law \( CC_{\text{circumcenter}} \) achieves \( T_{\epsilon}\text{-rendezvous} \).

Furthermore,

1. if any two agents belong to the same connected component at \( \ell \in \mathbb{N}_0 \), then they continue to belong to the same connected component subsequently; and
2. for each evolution, there exists \( P^* = (p^*_1, \ldots, p^*_n) \in (\mathbb{R}^d)^n \) such that:
   - the evolution asymptotically approaches \( P^* \), and
   - for each \( i, j \in \{1, \ldots, n\} \), either \( p^*_i = p^*_j \), or \( \|p^*_i - p^*_j\|_2 > r \).

Similar result for visibility networks in non-convex environments.

Theorem (Time complexity of circumcenter laws)

For \( r \in \mathbb{R}_{>0} \) and \( \epsilon \in (0, 1] \), the following statements hold:

1. on the network \( S_{\text{disk}} \), evolving on the real line \( \mathbb{R} \) (i.e., with \( d = 1 \)),
   \[ TC(T_{\text{rendezvous}}, CC_{\text{circumcenter}}) \in \Theta(n) \];
2. on the network \( S_{\text{LD}} \), evolving on the real line \( \mathbb{R} \) (i.e., with \( d = 1 \)),
   \[ TC(T_{\epsilon}\text{-rendezvous}, CC_{\text{circumcenter}}) \in \Theta(n^2 \log(nc^{-1})) \]; and

Similar results for visibility networks.

Robustness of circumcenter algorithms

Push whole idea further!, e.g., for robustness against link failures.

Look at evolution under link failures as outcome of nondeterministic evolution under multiple interaction topologies.

\[ P \rightarrow \{\text{evolution under } G_1, \text{evolution under } G_2, \text{evolution under } G_3\} \]
Fig. 2.2 A top-view photograph, using a polarizing filter, of the territories of the male Tilapia mossambica. Photograph and caption reprinted from G. W. Barlow, Hexagonal territories, Animal Behavior, Volume 22, 1974, by permission of Academic Press.

As an example of synchronous settling for which the territories can be visualized, Tilapia mossambica; the rim of the pits, form a pattern of polygons. The breeding males are the black fish, which are nonbreeding males. The fish with a conspicuous spot in its tail, in the upper-right corner, is the male. The males establish their territories, i.e., the locations at which they will spit sand away from the pit centers toward their neighbors. For a high enough density of fish, this reciprocal spitting was observed in a controlled experiment. Fish were introduced into a large outdoor pool with a uniformly sandy bottom. After the fish had established their territories, i.e., the locations at which they will spit sand, the territories were photographed. In Figure 2.2, the resulting photograph from [3] is reproduced. The territories are seen to be polygonal, and in [27, 59], it was shown that the territories must change shape and size as the fish spit sand from different locations.

Since all the fish are assumed to be of equal strength, i.e., they all presumably have equal reproductive potential, the desire to be as far away as possible from their neighbors leads to the formation of polygonal territories. The territories are seen to be polygonal, and in [27, 59], it was shown that the territories must change shape and size as the fish spit sand from different locations.

The fish, in their desire to be as far away as possible from their neighbors, tend to move towards the center of their territories, i.e., the locations at which they will spit sand. This results in the territories being polygonal, and in [27, 59], it was shown that the territories must change shape and size as the fish spit sand from different locations.

The territories can be modeled as Voronoi polyhedra, which are generated by the spitting behavior of the fish. The Voronoi partition $V(P) = \{V_1, \ldots, V_n\}$ generated by $(p_1, \ldots, p_n)$ is defined as:

$$V_i = \{q \in Q \mid \|q - p_i\| \leq \|q - p_j\|, \forall j \neq i\}$$

where $\mathcal{H}(p_i, p_j)$ is the half plane $(p_i, p_j)$.
Expected-value multicenter function

**Objective:** Given sensors/nodes/robots/sites \((p_1, \ldots, p_n)\) moving in environment \(Q\) achieve **optimal coverage**

\[
\phi : \mathbb{R}^d \to \mathbb{R}_{\geq 0} \text{ density}
\]

\[
f : \mathbb{R}_{\geq 0} \to \mathbb{R} \text{ non-increasing and piecewise continuously differentiable, possibly with finite jump discontinuities}
\]

maximize \(H_{\text{exp}}(p_1, \ldots, p_n) = E_{\phi} \left[ \max_{i \in \{1, \ldots, n\}} f(\|q - p_i\|) \right] \)

**H_{\text{exp}}-optimality of the Voronoi partition**

\(H_{\text{exp}}\) as a function of agent positions and partition,

\[
H_{\text{exp}}(p_1, \ldots, p_n, W_1, \ldots, W_n) = \sum_{i=1}^{n} \int_{W_i} f(\|q - p_i\|) \phi(q) dq
\]

Variety of scenarios

Alternative expression in terms of Voronoi partition,

\[
H_{\text{exp}}(p_1, \ldots, p_n) = \sum_{i=1}^{n} \int_{V_i(P)} f(\|q - p_i\|) \phi(q) dq
\]

**Distortion problem:** \(f(x) = -x^2\) gives rise to \((J_\phi(W, p)\) is moment of inertia)

\[
H_{\text{dist}}(p_1, \ldots, p_n) = -\sum_{i=1}^{n} J_\phi(V_i(P), p_i)
\]

**Area problem:** \(f(x) = 1_{[0,a]}(x), a \in \mathbb{R}_{>0} \) gives rise to

\[
H_{\text{area,a}}(p_1, \ldots, p_n) = \sum_{i=1}^{n} \text{area}_\phi(V_i(P) \cap \overline{B}(p_i, a))
\]

Using parallel axis theorem,

\[
H_{\text{dist}}(B(p_i, CM_\phi(W_i)), \ldots, CM_\phi(W_n), W_1, \ldots, W_n) = -\sum_{i=1}^{n} J_\phi(W_i, p_i)
\]

**Proposition (For fixed positions, Voronoi is optimal)**

Let \(P = \{p_1, \ldots, p_n\} \in F(S)\). For any performance function \(f\) and for any partition \(\{W_1, \ldots, W_n\} \subset \mathcal{P}(S)\) of \(S\),

\[
H_{\text{exp}}(p_1, \ldots, p_n, V_1(P), \ldots, V_n(P)) \geq H_{\text{exp}}(p_1, \ldots, p_n, W_1, \ldots, W_n),
\]

and the inequality is strict if any set in \(\{W_1, \ldots, W_n\}\) differs from the corresponding set in \(\{V_1(P), \ldots, V_n(P)\}\) by a set of positive measure

**Proposition**

Let \(\{W_1, \ldots, W_n\} \subset \mathcal{P}(S)\) be a partition of \(S\). Then,

\[
H_{\text{dist}}(CM_\phi(W_1), \ldots, CM_\phi(W_n), W_1, \ldots, W_n) \geq H_{\text{dist}}(p_1, \ldots, p_n, W_1, \ldots, W_n),
\]

and the inequality is strict if there exists \(i \in \{1, \ldots, n\}\) for which \(W_i\) has non-vanishing area and \(p_i \neq CM_\phi(W_i)\).
Gradient of $H_{\text{exp}}$ is distributed

For $f$ smooth

$$\frac{\partial H_{\text{exp}}}{\partial p_i}(P) = \int_{V_i(P)} \frac{\partial f}{\partial p_i}(\|q - p_i\|) \phi(q) dq$$

$$+ \int_{\partial V_i(P)} f(\|q - p_i\|) \langle n_i(q), \frac{\partial q}{\partial p_i} \rangle \phi(q) dq$$

$$+ \sum_{j \text{ neigh } i} \int_{V_i(P) \cap V_i(P)} f(\|q - p_j\|) \langle n_j(q), \frac{\partial q}{\partial p_i} \rangle \phi(q) dq$$

contrib from neighbors

Smoothness properties of $H_{\text{exp}}$

**Dscn($f$) (finite) discontinuities of $f$**

For $f$ smooth

$$\frac{\partial H_{\text{exp}}}{\partial p_i}(P) = \int_{V_i(P)} \frac{\partial}{\partial p_i} f(\|q - p_i\|) \phi(q) dq$$

$$+ \int_{\partial V_i(P)} f(\|q - p_i\|) \langle n_i(q), \frac{\partial q}{\partial p_i} \rangle \phi(q) dq$$

$$- \int_{\partial V_i(P)} f(\|q - p_i\|) \langle n_i(q), \frac{\partial q}{\partial p_i} \rangle \phi(q) dq$$

Therefore,

$$\frac{\partial H_{\text{exp}}}{\partial p_i}(P) = \int_{V_i(P)} \frac{\partial}{\partial p_i} f(\|q - p_i\|) \phi(q) dq$$

Particular gradients

**Distortion problem:** continuous performance,

$$\frac{\partial H_{\text{dist}}}{\partial p_i}(P) = 2 \text{ area}_\phi(V_i(P)) (\text{CM}_\phi(V_i(P)) - p_i)$$

**Area problem:** performance has single discontinuity,

$$\frac{\partial H_{\text{area},a}}{\partial p_i}(P) = \int_{V_i(P) \cap \partial B(p_i,a)} n_{\text{out},B(p_i,a)}(q) \phi(q) dq$$

Theorem

Expected-value multicenter function $H_{\text{exp}}: S^n \to \mathbb{R}$ is

- globally Lipschitz on $S^n$; and
- continuously differentiable on $S^n \setminus S_{\text{coinc}}$, where

$$\frac{\partial H_{\text{exp}}}{\partial p_i}(P) = \int_{V_i(P)} \frac{\partial}{\partial p_i} f(\|q - p_i\|_2) \phi(q) dq$$

$$+ \sum_{a \in \text{Dscn}(f)} (f_-(a) - f_+(a)) \int_{V_i(P) \cap \partial B(p_i,a)} n_{\text{out},B(p_i,a)}(q) \phi(q) dq$$

Therefore, the gradient of $H_{\text{exp}}$ is spatially distributed over $G_D$
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Geometric-center laws

Uniform networks \( S_D \) and \( S_{LD} \) of locally-connected first-order agents in a polytope \( Q \subset \mathbb{R}^d \) with the Delaunay and \( r \)-limited Delaunay graphs as communication graphs

All laws share similar structure

At each communication round each agent performs:

- it transmits its position and receives its neighbors’ positions;
- it computes a notion of geometric center of its own cell determined according to some notion of partition of the environment

Between communication rounds, each robot moves toward this center

VRN-CNTRD ALGORITHM
Optimizes distortion \( \mathcal{H}_{dist} \)

Robotic Network: \( S_D \) in \( Q \), with absolute sensing of own position

Distributed Algorithm: VRN-CNTRD

Alphabet: \( L = \mathbb{R}^d \cup \{ \text{null} \} \)

function \( \text{msg}(p, i) \)
1: return \( p \)

defunction \( \text{ctrl}(p, y) \)
1: \( V := Q \cap (\bigcap \{ H_{p, p_{rcvd}} \mid \text{for all non-null } p_{rcvd} \in y \} ) \)
2: return \( \text{CM}_\phi(V) - p \)

Simulation

For \( \epsilon \in \mathbb{R}_{>0} \), the \( \epsilon \)-distortion deployment task

\[
\mathcal{T}_{\epsilon\text{-distor-dply}}(P) = \begin{cases} 
\text{true}, & \text{if } ||p[i] - \text{CM}_\phi(V[i])(p)||_2 \leq \epsilon, i \in \{1, \ldots, n\}, \\
\text{false}, & \text{otherwise}, 
\end{cases}
\]
Voronoi-centroid law on planar vehicles

Robotic Network: $S_{\text{vehicles}}$ in $Q$ with absolute sensing of own position

Distributed Algorithm: $\text{VRN-CNTRD-DYNMCS}$

Alphabet: $L = \mathbb{R}^2 \cup \text{null}$

function $\text{msg}(p, \theta, i)$

1: return $p$

function $\text{ctrl}(p, \theta, (p_{\text{smpld}}, \theta_{\text{smpld}}), y)$

1. $V := Q \cap \left( \bigcap \{ H_{p_{\text{smpld}}, p_{\text{rcvd}}} \mid \text{for all non-null } p_{\text{rcvd}} \in y \} \right)$
2. $v := -k_p \text{prop}(\cos \theta, \sin \theta) \cdot (p - CM_\phi(V))$
3. $\omega := 2k_p \text{prop} \arctan \left( \frac{-\sin \theta, \cos \theta \cdot (p - CM_\phi(V))}{\cos \theta, \sin \theta \cdot (p - CM_\phi(V))} \right)$
4. return $(v, \omega)$

Algorithm illustration

Lmtd-Vrn-nrml algorithm

Optimizes area $H_{\text{area}, \frac{\pi}{2}}$

Robotic Network: $S_{\text{LD}}$ in $Q$ with absolute sensing of own position and with communication range $r$

Distributed Algorithm: $\text{LMTD-VRN-NRML}$

Alphabet: $L = \mathbb{R}^d \cup \{\text{null}\}$

function $\text{msg}(p, i)$

1: return $p$

function $\text{ctrl}(p, y)$

1. $V := Q \cap \left( \bigcap \{ H_{p_{\text{rcvd}}} \mid \text{for all non-null } p_{\text{rcvd}} \in y \} \right)$
2. $v := \int_{V \cap B(p, \frac{\pi}{2})} n_{\text{out}, B(p, \frac{\pi}{2})}(q) \phi(q) dq$
3. $\lambda_s := \max \left\{ \lambda \mid \delta \mapsto \int_{V \cap B(p + \delta v, \frac{\pi}{2})} \phi(q) dq \text{ is strictly increasing on } [0, \lambda] \right\}$
4. return $\lambda_s v$

Simulation

initial configuration | gradient descent | final configuration
Simulation

For $r, \epsilon \in \mathbb{R}^+$,
\[ T_{\epsilon-r\text{-area-dply}}(P) = \begin{cases} 
\text{true}, & \| \int_{V_{\{i\}}(P) \cap \partial B(p_{\{i\}}, r)} \mathbb{B}(p_{\{i\}}, r)(q) \phi(q) dq \|_2 \leq \epsilon, \quad i \in \{1, \ldots, n\}, \\
\text{false}, & \text{otherwise}.
\end{cases} \]

Correctness of the geometric-center algorithms

Theorem

For $d \in \mathbb{N}$, $r \in \mathbb{R}^+$ and $\epsilon \in \mathbb{R}^+$, the following statements hold.

1. on the network $S_D$, the law $CC_{\text{VRN-CNTRD}}$ achieves the $\epsilon$-distortion deployment task $T_{\epsilon\text{-distor-dply}}$. Moreover, any execution monotonically optimizes $H_{\text{dist}}^2$.

2. on the network $S_{\text{vehicles}}$, the law $CC_{\text{VRN-CNTRD-DYMCOS}}$ achieves the $\epsilon$-distortion deployment task $T_{\epsilon\text{-distor-dply}}$. Moreover, any execution monotonically optimizes $H_{\text{dist}}^3$.

3. on the network $S_{\text{LD}}$, the law $CC_{\text{LMTD-VRN-NRML}}$ achieves the $\epsilon$-r-area deployment task $T_{\epsilon-r\text{-area-dply}}$. Moreover, any execution monotonically optimizes $H_{\text{area}, r}$.

Time complexity of $CC_{\text{LMTD-VRN-CNTRD}}$

Assume $\text{diam}(Q)$ is independent of $n$, $r$ and $\epsilon$

Theorem (Time complexity of LMTD-VRN-CNTRD law)

Assume the robots evolve in a closed interval $Q \subset \mathbb{R}$, that is, $d = 1$, and assume that the density is uniform, that is, $\phi \equiv 1$. For $r \in \mathbb{R}^+$ and $\epsilon \in \mathbb{R}^+$, on the network $S_{\text{LD}}$

\[ TC(T_{\epsilon-r\text{-distor-area-dply}}, CC_{\text{LMTD-VRN-CNTRD}}) \in O(n^3 \log(n\epsilon^{-1})) \]

Open problem: characterize complexity of deployment algorithms in higher dimensions

Outline

1. Models for multi-agent networks
2. Rendezvous and connectivity maintenance
   - Maintaining connectivity
   - Circumcenter algorithms
   - Correctness analysis via nondeterministic systems
3. Deployment
   - Expected-value deployment
   - Geometric-center laws
   - Disk-covering and sphere-packing deployment
4. Synchronized boundary patrolling
   - Balanced synchronization
   - Unbalanced synchronization
5. Conclusions
Deployment: basic behaviors

“move away from closest”

“move towards furthest”

Equilibria? Asymptotic behavior?
Optimizing network-wide function?

Deployment: 1-center optimization problems

\[ \text{sm}_Q(p) = \min \{ \| p - q \| \mid q \in \partial Q \} \]

Lipschitz \( 0 \in \partial \text{sm}_Q(p) \iff p \in \text{IC}(Q) \)

\[ \text{lg}_Q(p) = \max \{ \| p - q \| \mid q \in \partial Q \} \]

Lipschitz \( 0 \in \partial \text{lg}_Q(p) \iff p \in \text{CC}(Q) \)

Locally Lipschitz function \( V \) are differentiable a.e.

Generalized gradient of \( V \) is

\[ \partial V(x) = \text{convex closure} \{ \text{lim}_{i \to \infty} \nabla V(x_i) \mid x_i \to x, x_i \not\in \Omega_V \cup S \} \]

Nonsmooth LaSalle Invariance Principle

Evolution of \( V \) along Filippov solution \( t \mapsto V(x(t)) \) is differentiable a.e.

\[ \frac{d}{dt} V(x(t)) \in \tilde{L}_X V(x(t)) = \{ a \in \mathbb{R} \mid \exists v \in K[X](x) \text{ s.t. } \zeta \cdot v = a, \forall \zeta \in \partial V(x) \} \]

set-valued Lie derivative

LaSalle Invariance Principle

For \( S \) compact and strongly invariant with \( \max \tilde{L}_X V(x) \leq 0 \)

Any Filippov solution starting in \( S \) converges to largest weakly invariant set contained in \( \{ x \in S \mid 0 \in \tilde{L}_X V(x) \} \)

E.g., nonsmooth gradient flow \( \dot{x} = -\text{Ln}[\partial V](x) \) converges to critical set
Deployment: multi-center optimization
sphere packing and disk covering
“move away from closest”: \[
\dot{p}_i = + \ln(\partial \text{sm}_{V_i(P)}(p_i)) - \text{at fixed } V_i(P)
\]
“move towards furthest”: \[
\dot{p}_i = - \ln(\partial \text{lg}_{V_i(P)}(p_i)) - \text{at fixed } V_i(P)
\]

Aggregated objective functions:
\[
\mathcal{H}_{sp}(P) = \min_i \text{sm}_{V_i(P)}(p_i) = \min_i \left[ \frac{1}{2} \| p_i - p_j \|, \text{dist}(p_i, \partial Q) \right]
\]
\[
\mathcal{H}_{dc}(P) = \max_i \text{lg}_{V_i(P)}(p_i) = \max_q \left[ \min_i \| q - p_i \| \right]
\]

Critical points of \(\mathcal{H}_{sp}\) and \(\mathcal{H}_{dc}\) (locally Lipschitz)
- If \(0 \in \text{int} \partial \mathcal{H}_{sp}(P)\), then \(P\) is strict local maximum, all agents have same cost, and \(P\) is incenter Voronoi configuration
- If \(0 \in \text{int} \partial \mathcal{H}_{dc}(P)\), then \(P\) is strict local minimum, all agents have same cost, and \(P\) is circumcenter Voronoi configuration

Aggregate functions monotonically optimized along evolution
\[
\min \tilde{\mathcal{L}} \ln(\partial \text{sm}_{V_i(P)}(P)) \mathcal{H}_{sp}(P) \geq 0
\]
\[
\max \tilde{\mathcal{L}} - \ln(\partial \text{lg}_{V_i(P)}(P)) \mathcal{H}_{dc}(P) \leq 0
\]

Asymptotic convergence via nonsmooth LaSalle principle
- Convergence to configurations where all agents whose local cost coincides with aggregate cost are centered
- Convergence to center Voronoi configurations still open

Voronoi-circumcenter algorithm

Robotic Network: \(S_D\) in \(Q\) with absolute sensing of own position
Distributed Algorithm: VRN-CRCMCNTR
Alphabet: \(L = \mathbb{R}^d \cup \{\text{null}\}\)
function msg\((p, i)\)
1: return \(p\)
function ctrl\((p, y)\)
1: \(V := Q \cap \left( \bigcap \{H_{p, prvcd} \mid \text{for all non-null } p_{prvd} \in y\} \right)\)
2: return \(\text{CC}(V) - p\)

Voronoi-incenter algorithm

Robotic Network: \(S_D\) in \(Q\) with absolute sensing of own position
Distributed Algorithm: VRN-NCNTR
Alphabet: \(L = \mathbb{R}^d \cup \{\text{null}\}\)
function msg\((p, i)\)
1: return \(p\)
function ctrl\((p, y)\)
1: \(V := Q \cap \left( \bigcap \{H_{p, prvcd} \mid \text{for all non-null } p_{prvd} \in y\} \right)\)
2: return \(x \in \text{IC}(V) - p\)
For $\epsilon \in \mathbb{R} > 0$, the $\epsilon$-disk-covering deployment task

$$T_{\epsilon\text{-dc-dply}}(P) = \begin{cases} 
  \text{true}, & \text{if } \|p[i] - \text{CC}(V[i](P))\|_2 \leq \epsilon, \ i \in \{1, \ldots, n\}, \\
  \text{false}, & \text{otherwise},
\end{cases}$$

For $\epsilon \in \mathbb{R} > 0$, the $\epsilon$-sphere-packing deployment task

$$T_{\epsilon\text{-sp-dply}}(P) = \begin{cases} 
  \text{true}, & \text{if } \text{dist}_2(p[i], \text{IC}(V[i](P))) \leq \epsilon, \ i \in \{1, \ldots, n\}, \\
  \text{false}, & \text{otherwise},
\end{cases}$$

Theorem

For $d \in \mathbb{N}$, $r \in \mathbb{R} > 0$ and $\epsilon \in \mathbb{R} > 0$, the following statements hold.

1. on the network $S_D$, any execution of the law $\text{CC}_{\text{VRN-CRCMCNTR}}$ monotonically optimizes the multicenter function $H_{dc}$;
2. on the network $S_D$, any execution of the law $\text{CC}_{\text{VRN-NCNTR}}$ monotonically optimizes the multicenter function $H_{sp}$.

Synchronized boundary patrolling

Joint work with Sara Susca (Honeywell) and Sonia Martínez (UCSD)

1. some UAVs move along boundary of sensitive territory
2. short-range communication and sensing
3. surveillance objective: minimize service time for appearing events communication network connectivity

Example motion:

![Example motion diagram](image)

Analogy with mechanics and dynamics

1. robots with “communication impacts” analogous to beads on a ring
2. classic subject in dynamical systems and geometric mechanics: billiards, iterated impact dynamics, gas theory of hard spheres
3. rich dynamics with even just 3 beads (distinct masses, elastic collisions) dynamics akin billiard flow inside acute triangle dense periodic and nonperiodic modes, chaotic collision sequences

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* Iterated Impact Dynamics of $N$-Beads on a Ring*

Bryan Cooley1
Paul K. Newton1

**Abstract.** When $N$-beads slide along a frictionless hoop, their collision sequence gives rise to a dynamical system that can be studied via matrix products. It is of general interest to understand the distribution of values and the corresponding eigenvalue spectrum that a given collision sequence can produce. We formulate the problem for general $N$ and state some basic theorems regarding the eigenvalues of the collision matrices and their products. The
Boundary patrolling: synchronized bead oscillation

Desired synchronized behavior:
- starting from random initial positions and velocities
- every bead impacts its neighbor at the same point
- simultaneous impacts

Design specification for synchronization algorithm

Achieve: asymptotically stabilize synchronized motion
Subject to:
- arbitrary initial positions, velocities and directions of motion
- beads can measure traveled distance, however no absolute localization capability, no knowledge of circle length
- no knowledge about \( n \), adaptation to changing \( n \) (even and odd)
- anonomous agents with memory and message sizes independent of \( n \)
- smooth dependency upon effect of measurement and control noise

Fully-adaptive feedback synchronization

Slowdown-Impact-Speedup algorithm

Algorithm: (for presentation’s sake, beads sense their position)

1st phase: compute average speed \( v \) and desired sweeping arcs

2nd phase for \( f \in \frac{1}{2}, 1 \), each bead:
- moves at nominal speed \( v \) if inside its desired sweeping arc
- slows down \( (fv) \) when moving away of its sweeping arc
- hesitate when early
- when impact, change direction
- speeds up when moving towards its desired sweeping arc

Simulations results: balanced synchronization

Balanced initial condition:
- \( n \) is even
- \( d_i \) is direction of motion
- \( \sum_i^n d_i(0) = \sum_i^n d_i(t) = 0 \)
- \( n/2 \) move initially clockwise
First phase: average speed and sweeping arc

If an impact between bead $i$ and $i+1$ occurs:
- beads average nominal speeds: $v_i^+ = v_{i+1}^+ = 0.5(v_i + v_{i+1})$
- beads change their direction of motion if $d_i = -d_{i+1}$ (head-head type)
- beads update their desired sweeping arc

Challenges

1. how to prove balanced synchronization?
2. what happens for unbalanced initial conditions $\sum_i^n d_i(0) \neq 0$?
3. what happens for $n$ is odd?
4. how to describe the system with a single variable?

Modeling detour

- configuration space
  - order-preserving dynamics $\theta_i \in \text{Arc}(\theta_{i-1}, \theta_{i+1})$ on $\mathbb{T}^n$
  - $\Delta^n \times \{c, cc\}^n \times (\mathbb{R}^>)^n \times \{\text{arcs}\}^n \times \{\text{away, towards}\}^n$
- hybrid system with
  - piecewise constant dynamics
  - event-triggered jumps: impact, cross boundary

Passage and return times

- passage time: $t_k^i = k$th time when bead $i$ passes by sweeping arc center
- return time: $\delta_i(t) = \text{duration between last two passage times}$
- if impact between beads $(i, i+1)$ at time $t$, then
  $$\begin{bmatrix} \delta_i \\ \delta_{i+1} \end{bmatrix}(t^+) = \begin{bmatrix} \frac{1-t}{1+t} & \frac{2t}{1+t} \\ \frac{2t}{1+t} & \frac{1-t}{1+t} \end{bmatrix} \begin{bmatrix} \delta_i \\ \delta_{i+1} \end{bmatrix}(t^-)$$

exponential average consensus
Averaging algorithms

Distributed averaging algorithm or consensus algorithms

\[ x(\ell + 1) = Ax(\ell) \]

with (row) stochastic matrix \( A \): \( \sum_{j=1}^{n} a_{ij} = 1 \) and \( a_{ij} \geq 0 \)

- let \( G(A) \) be unweighted matrix associated to \( A \)
- a sequence of stochastic \( \{A(\ell)\}_{\ell \in \mathbb{N}} \) is non-degenerate if \( \exists \alpha > 0 \) s.t.
  \( a_{ii}(\ell) \geq \alpha \) and \( a_{ij}(\ell) \in \{0\} \cup [\alpha, 1] \), for all \( i \neq j \)

**Theorem (Convergence to average consensus)**

Let \( \{A(\ell)\}_{\ell \in \mathbb{N}} \) be a non-degenerate sequence of stochastic, symmetric matrices
- each evolution \( x \) converges to average(\( x(0) \))\( 1_n \)
- for all \( \ell \in \mathbb{N} \), the graph \( \bigcup_{\tau \geq \ell} G(A(\tau)) \) is connected

### Convergence results: balanced synchronization

Balanced Synchronization Theorem: For balanced initial directions, assume
- exact average speed and desired sweeping arcs
- initial conditions lead to well-defined 1st passage times
Then balanced synchronization is asymptotically stable

\[ \lim_{t \to \infty} \delta(t) = \frac{2\pi}{Nv} 1_n, \quad \lim_{k \to +\infty} \| T^k - \frac{1_n \cdot T^k}{n} 1_n \| = 0 \]

### Conjectures arising from simulation results

Only assumption required is balanced initial conditions.
- analysis of cascade consensus algorithms
- global attractivity of synchronous behavior

### Simulations results: 1-unbalanced case

1-unbalanced initial condition:
- \( n \) is odd
- \( \sum_{i=1}^{n} d_i(0) = \sum_{i=1}^{n} d_i(t) = \pm 1 \)
**1-unbalanced synchronization**

- $f \in \left[ \frac{1}{2}, \frac{n}{1+n} \right]$
- 1-unbalanced sync: beads meet at arcs boundaries $\pm \frac{2\pi}{n^2} \frac{f}{1-f}$

1-unbalanced Synchronization Theorem: For $\sum^n_i d_i(0) = \pm 1$, assume
- exact average speed and desired sweeping arcs
- initial conditions lead to well-defined 1st passage times

Then 1-unbalanced synchronization is asymptotically stable

$$\lim_{t \to \infty} \delta(t) = \frac{2\pi}{Nv} \mathbf{1}_n, \quad \lim_{k \to +\infty} \left( T^{2k} - T^{2(k-1)} \right) = \frac{2}{v} \frac{2\pi}{n} \mathbf{1}_n$$

**General unbalanced case**

Conjecture global asy-synchronization in the balanced and unbalanced case

$D$-unbalanced period orbits Theorem:
Let $\sum^n_i d_i(0) = \pm D$. If there exists an orbit along which beads $i$ and $i+1$ meet at boundary $\pm \frac{2\pi}{n^2} \frac{f}{1-f}$, then $f < \frac{n/|D|}{1 + n/|D|}$.

**Summary and conclusions**

Examined various motion coordination tasks
- **rendezvous**: circumcenter algorithms
- **connectivity maintenance**: flexible constraint sets in convex/nonconvex scenarios
- **deployment**: gradient algorithms based on geometric centers
- **beads problem**: robotic patrolling via synchronization

Correctness and (1-d) complexity analysis of geometric-center control and communication laws via
- Discrete- and continuous-time nondeterministic dynamical systems
- Invariance principles, stability analysis
- Geometric structures and geometric optimization

 Plenty of open problems!

**Motion coordination is emerging discipline**

Literature is full of exciting problems, solutions, and tools we have not covered

Formation control, consensus, cohesiveness, flocking, collective synchronization, boundary estimation, cooperative control over constant graphs, quantization, asynchronism, delays, distributed estimation, spatial estimation, data fusion, target tracking, networks with minimal capabilities, target assignment, vehicle dynamics and energy-constrained motion, vehicle routing, dynamic servicing problems, load balancing, robotic implementations,...

Too long a list to fit it here!