consider the mouthbreeder fish excavate breeding pits in sandy bottoms by spitting sand away from the pit centers. This results in sand parapets that are visible territorial boundaries. In [3], the results of a controlled experiment were given. Fish were introduced into a large outdoor pool with a uniform sandy bottom. After the fish had established their territories, i.e., the locations at which they will spit sand, the pit centers as far away as possible from their neighbors cause the fish to continuously adjust the position of the pit centers. This adjustment process is modeled as follows.

Since all the fish share assumed to be of equal strength, i.e., they all presumably have a similar desire to place their spitting location toward the centroid of their current territory; subsequently, the parapets separating territories must change since the fish spit from different locations.

The territories were photographed in Figure 2.2. A top-view photograph, using a polarizing filter, of the territories of the male Tilapia mossambica; the rim of the pits, form a pattern of polygons. The breeding males are the black fish, which are the territorial males of this species. Territorial males of this species range in size from about 20 cm to 200 cm. The gray fish share the females, juveniles, and non-breeding males.

The fish with a conspicuous spot in its tail, in the upper-right corner, is the breeding male, and it is the only fish that can fertilize the eggs of the females. The breeding shakes, which are males with a red spot on their tail, are the non-breeding males. The territories were photographed in Figure 2.2. The results of [22, 60] show that the territories are polygonal and, in [27, 59], it was shown that they are very closely approximated by a Voronoi tessellation.

Another motion coordination objective: deployment

Objective: Given sensors/nodes/robots/sites (p_1, ..., p_n) moving in environment Q achieve optimal coverage

\[ \phi : \mathbb{R}^d \rightarrow \mathbb{R}_{\geq 0} \text{ density} \]

\[ f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R} \text{ non-increasing and piece-wise continuously differentiable, possibly with finite jump discontinuities} \]

maximize \[ H_{\text{exp}}(p_1, ..., p_n) = E_{\phi} \left[ \max_{i \in \{1, ..., n\}} f(\|q - p_i\|) \right] \]
**Proposition**

Let $P = \{p_1, \ldots, p_n\} \in \mathbb{P}(S)$. For any performance function $f$ and for any partition $\{W_1, \ldots, W_n\} \subset \mathbb{P}(S)$ of $S$,

$$H_{\text{exp}}(p_1, \ldots, p_n, V_1(P), \ldots, V_n(P)) \geq H_{\text{exp}}(p_1, \ldots, p_n, W_1, \ldots, W_n),$$

and the inequality is strict if any set in $\{W_1, \ldots, W_n\}$ differs from the corresponding set in $\{V_1(P), \ldots, V_n(P)\}$ by a set of positive measure.

**Area problem**

$f(x) = 1_{[0,a]}(x), \ a \in \mathbb{R}_{>0}$

$$H_{\text{area}, a}(p_1, \ldots, p_n) = \sum_{i=1}^{n} \int_{V_i(P)} 1_{[0,a]}(\|q - p_i\|_2) \phi(q) dq$$

$$= \sum_{i=1}^{n} \int_{V_i(P) \cap \overline{B}(p_i, a)} \phi(q) dq$$

$$= A_\phi(V_i(P) \cap \overline{B}(p_i, a)) = A_\phi(\bigcup_{i=1}^{n} \overline{B}(p_i, a)),$$

Area, measured according to $\phi$, covered by the union of the $n$ balls $\overline{B}(p_1, a), \ldots, \overline{B}(p_n, a)$.

**Mixed distortion-area problem**

$f(x) = -x^2 \ 1_{[0,a]}(x) + b \cdot 1_{a,+\infty}(x), \ a \in \mathbb{R}_{>0}$ and $b \leq a^2$

$$H_{\text{distor-area}, a,b}(p_1, \ldots, p_n) = - \sum_{i=1}^{n} J_\phi(V_i(P), p_i) + b A_\phi(Q \setminus \bigcup_{i=1}^{n} \overline{B}(p_i, a)),$$

If $b = a^2$, performance $f$ is continuous, and we write $\mathcal{H}_{\text{distor-area}, a}$. Extension to sets of points and partitions reads

$$H_{\text{distor-area}, a}(p_1, \ldots, p_n, W_1, \ldots, W_n)$$

$$= - \sum_{i=1}^{n} \left( J_\phi(W_i \cap \overline{B}(p_i, a), p_i) + a^2 A_\phi(W_i \cap (S \setminus \overline{B}(p_i, a))) \right).$$

**Proposition** ($\mathcal{H}_{\text{distor-area}, a}$-optimality of centroid locations)

Let $\{W_1, \ldots, W_n\} \subset \mathbb{P}(S)$ be a partition of $S$. Then,

$$H_{\text{distor-area}, a}(CM_\phi(W_1 \cap \overline{B}(p_1, a)), \ldots, CM_\phi(W_n \cap \overline{B}(p_n, a)), W_1, \ldots, W_n)$$

$$\geq H_{\text{distor}}(p_1, \ldots, p_n, W_1, \ldots, W_n),$$

and the inequality is strict if there exists $i \in \{1, \ldots, n\}$ for which $W_i$ has non-vanishing area and $p_i \neq CM_\phi(W_i \cap \overline{B}(p_i, a))$.

**Distortion problem**

$f(x) = -x^2$

$$H_{\text{distor}}(p_1, \ldots, p_n) = - \sum_{i=1}^{n} \int_{V_i(P)} \|q - p_i\|_2^2 \phi(q) dq = - \sum_{i=1}^{n} J_\phi(V_i(P), p_i)$$

($J_\phi(W, p)$ is moment of inertia). Note

$$H_{\text{distor}}(p_1, \ldots, p_n, W_1, \ldots, W_n)$$

$$= - \sum_{i=1}^{n} J_\phi(W_i, CM_\phi(W_i)) - \sum_{i=1}^{n} A_\phi(W_i) \|p_i - CM_\phi(W_i)\|_2^2$$

**Proposition**

Let $\{W_1, \ldots, W_n\} \subset \mathbb{P}(S)$ be a partition of $S$. Then,

$$H_{\text{distor}}(CM_\phi(W_1), \ldots, CM_\phi(W_n), W_1, \ldots, W_n)$$

$$\geq H_{\text{distor}}(p_1, \ldots, p_n, W_1, \ldots, W_n),$$

and the inequality is strict if there exists $i \in \{1, \ldots, n\}$ for which $W_i$ has non-vanishing area and $p_i \neq CM_\phi(W_i)$.
Smoothness properties of $\mathcal{H}_{\text{exp}}$

Dscn($f$) (finite) discontinuities of $f$

Consider the case of smooth performance $f$,

$$
\mathcal{H}_{\text{exp}}(P) = \int_{V_i(P)} \frac{\partial}{\partial p_i} f(\|q - p_i\|) \phi(q) dq
$$

Some proof ideas

Some proof ideas

Consider the case of smooth performance $f$,

$$
\mathcal{H}_{\text{area},a}(P) = \int_{V_i(P) \cap \partial B(p_i,a)} n_{\text{out},B(p_i,a)}(q) \phi(q) dq
$$

Particular gradients

Distortion problem: continuous performance,

$$
\mathcal{H}_{\text{distor}}(P) = 2 A \phi(V_i(P))(CM \phi(V_i(P)) - p_i)
$$

Area problem: performance has single discontinuity,

$$
\mathcal{H}_{\text{area},a}(P) = \int_{V_i(P) \cap \partial B(p_i,a)} n_{\text{out},B(p_i,a)}(q) \phi(q) dq
$$

Mixed distortion-area: continuous performance ($b = -a^2$),

$$
\mathcal{H}_{\text{distor-area},a}(P) = 2 A \phi(V_i,a(P))(CM \phi(V_i,a(P)) - p_i)
$$
Tuning the optimization problem

Gradients of $H_{\text{area},a}$, $H_{\text{distor-area},a,b}$ are distributed over $G_{\text{LD}}(2a)$
Robotic agents with range-limited interactions can compute gradients of $H_{\text{area},a}$ and $H_{\text{distor-area},a,b}$ as long as $r \geq 2a$

**Proposition (Constant-factor approximation of $H_{\text{distor}}$)**

Let $S \subset \mathbb{R}^d$ be bounded and measurable. Consider the mixed distortion-area problem with $a \in [0, \text{diam}(S)]$ and $b = -\text{diam}(S)^2$. Then, for all $P \in S^a$,

$$H_{\text{distor-area},a,b}(P) \leq H_{\text{distor}}(P) \leq \beta^2 H_{\text{distor-area},a,b}(P) < 0,$$

where $\beta = \frac{a}{\text{diam}(S)} \in [0, 1]$

Similarly, constant-factor approximations of $H_{\exp}$

**VRN-CNTRD ALGORITHM**
Optimizes distortion $H_{\text{distor}}$

Robotic Network: $S_D$ in $Q$, with absolute sensing of own position
Distributed Algorithm: VRN-CNTRD
Alphabet: $A = \mathbb{R}^d \cup \{\text{null}\}$
function msg($p$, $i$)
  1: return $p$
function ctrl($p$, $y$)
  1: $V := Q \cap (\bigcap \{H_{p,p_{\text{rcvd}}} | \text{for all non-null } p_{\text{rcvd}} \in y\})$
  2: return $\text{CM}_\phi(V) - p$

**Simulation**

For $\varepsilon \in \mathbb{R}_{>0}$, the $\varepsilon$-distortion deployment task

$$T_{\varepsilon}\text{-distor-dply}(P) = \begin{cases} \text{true}, & \text{if } \|p_i - \text{CM}_\phi(V_i(P))\|_2 \leq \varepsilon, \ i \in \{1, \ldots, n\}, \\ \text{false}, & \text{otherwise}, \end{cases}$$

Geometric-center laws

Uniform networks $S_D$ and $S_{LD}$ of locally-connected first-order agents in a polytope $Q \subset \mathbb{R}^d$ with the Delaunay and $r$-limited Delaunay graphs as communication graphs

All laws share similar structure
At each communication round each agent performs the following tasks:
- it transmits its position and receives its neighbors’ positions;
- it computes a notion of geometric center of its own cell determined according to some notion of partition of the environment

Between communication rounds, each robot moves toward this center
Voronoi-centroid law on planar vehicles

Robotic Network: $\mathcal{S}_{\text{vehicles}}$ in $Q$ with absolute sensing of own position

Distributed Algorithm: $\text{VRN-CNTRD-DYNMCS}$

Alphabet: $A = \mathbb{R}^2 \cup \{\text{null}\}$

function $\text{msg}(p, \theta, i)$
1: return $p$

function $\text{ctl}(p, \theta, (p_{\text{ampld}}, \theta_{\text{ampld}}), y)$
1: $V := Q \cap \left( \bigcap \{H_{p_{\text{ampld}}, p_{\text{rcvd}}} \mid \text{for all non-null } p_{\text{rcvd}} \in y \} \right)$
2: $v := -k_{\text{prop}} (\cos \theta, \sin \theta) \cdot (p - \text{CM}_\phi(V))$
3: $\omega := 2k_{\text{prop}} \arctan \left( \frac{-\sin \theta, \cos \theta}{\cos \theta, \sin \theta} \right) \cdot (p - \text{CM}_\phi(V))$
4: return $(v, \omega)$

Algorithm illustration

Simulation

LMTD-VRN-NRML algorithm
Optimizes area $H_{\text{area, } r^2}$

Robotic Network: $\mathcal{S}_{\text{LD}}$ in $Q$ with absolute sensing of own position and with communication range $r$

Distributed Algorithm: $\text{LMTD-VRN-NRML}$

Alphabet: $A = \mathbb{R}^d \cup \{\text{null}\}$

function $\text{msg}(p, i)$
1: return $p$

function $\text{ctl}(p, y)$
1: $V := Q \cap \left( \bigcap \{H_{p_{\text{rcvd}}} \mid \text{for all non-null } p_{\text{rcvd}} \in y \} \right)$
2: $v := \int_{V \cap \partial B(p, r^2)} \nabla_{out, B(p, r^2)}(q) \phi(q) dq$
3: $\lambda_* := \max \{ \lambda \mid \delta \mapsto \int_{V \cap \partial B(p + \delta v, r^2)} \phi(q) dq \text{ is strictly increasing on } [0, \lambda] \}$
4: return $\lambda_* v$
Simulation

For $r, \varepsilon \in \mathbb{R}_{>0}$,
\[
T_{\varepsilon-r-\text{area-dply}}(P) = \begin{cases} 
\text{true}, & \text{if } \|p[i] - \text{CM}_\phi(V[i](P))\|_2 \leq \varepsilon, \ i \in \{1, \ldots, n\} \\
\text{false}, & \text{otherwise.}
\end{cases}
\]

Optimizing $\mathcal{H}_{\text{distor}}$ via constant-factor approximation

LMTD-VRN-CNTRD algorithm
Optimizes $\mathcal{H}_{\text{distor-area, r}}$

Robotic Network: $\mathcal{S}_{\text{LD}}$ in $Q$ with absolute sensing of own position, and with communication range $r$

Distributed Algorithm: LMTD-VRN-CNTRD

Alphabet: $\mathbb{A} = \mathbb{R}^d \cup \{\text{null}\}$

function msg($p, i$)
1: return $p$

function ctl($p, y$)
1: $V := Q \cap B(p, r^2) \cap \cap \{H_{p, p}\text{rcvd} | \text{for all non-null} p_{\text{rcvd}} \in y\}$
2: return $\text{CM}_\phi(V) - p$

Limited range
run #1: 16 agents, density $\phi$ is sum of 4 Gaussians, time invariant, 1st order dynamics

Unlimited range
run #2: 16 agents, density $\phi$ is sum of 4 Gaussians, time invariant, 1st order dynamics
Correctness of the geometric-center algorithms

Theorem
For \( d \in \mathbb{N}, r \in \mathbb{R}^+ \) and \( \varepsilon \in \mathbb{R}^+ \), the following statements hold.

1. on the network \( S_D \), the law \( \text{CC}_\text{Vrn-cntrd} \) and on the network \( S_{\text{vehicles}} \), the law \( \text{CC}_\text{Vrn-cntrd-dynmcs} \) both achieve the \( \varepsilon \)-distortion deployment task \( T_{\varepsilon\text{-distor-dply}} \). Moreover, any execution of \( \text{CC}_\text{Vrn-cntrd} \) and \( \text{CC}_\text{Vrn-cntrd-dynmcs} \) monotonically optimizes the multicenter function \( H_{\text{distor}} \);

2. on the network \( S_{LD} \), the law \( \text{CC}_\text{Lmtd-Vrn-nrml} \) achieves the \( \varepsilon \)-\( r \)-area deployment task \( T_{\varepsilon\text{-}r\text{-area-dply}} \). Moreover, any execution of \( \text{CC}_\text{Lmtd-Vrn-nrml} \) monotonically optimizes the multicenter function \( H_{\text{area}, r} \);

3. on the network \( S_{LD} \), the law \( \text{CC}_\text{Lmtd-Vrn-cntrd} \) achieves the \( \varepsilon \)-\( r \)-distortion-area deployment task \( T_{\varepsilon\text{-}r\text{-distor-area-dply}} \). Moreover, any execution of \( \text{CC}_\text{Lmtd-Vrn-cntrd} \) monotonically optimizes the multicenter function \( H_{\text{distor-area}, r} \).

Time complexity of \( \text{CC}_\text{Lmtd-Vrn-cntrd} \)
Assume \( \text{diam}(Q) \) is independent of \( n, r \) and \( \varepsilon \)

Theorem (Time complexity of \( \text{Lmtd-Vrn-cntrd} \) law)
Assume the robots evolve in a closed interval \( Q \subset \mathbb{R} \), that is, \( d = 1 \), and assume that the density is uniform, that is, \( \phi \equiv 1 \). For \( r \in \mathbb{R}^+ \) and \( \varepsilon \in \mathbb{R}^+ \), on the network \( S_{LD} \)

\[
\text{TC}(T_{\varepsilon\text{-}r\text{-distor-area-dply}}, \text{CC}_\text{Lmtd-Vrn-cntrd}) \in O(n^3 \log(n\varepsilon^{-1}))
\]
Deployment: 1-center optimization problems

Evolution of $V$ along Filippov solution $t \mapsto V(x(t))$ is differentiable a.e.

$$
\frac{d}{dt} V(x(t)) \in \tilde{L}_X V(x(t)) = \{ a \in \mathbb{R} \mid \exists v \in K[X](x) \text{ s.t. } \zeta \cdot v = a, \forall \zeta \in \partial V(x) \}
$$

set-valued Lie derivative

LaSalle Invariance Principle

For $S$ compact and strongly invariant with $\max \tilde{L}_X V(x) \leq 0$

Any Filippov solution starting in $S$ converges to largest weakly invariant set contained in $\{ x \in S \mid 0 \in \tilde{L}_X V(x) \}$

E.g., nonsmooth gradient flow $\dot{x} = -\ln[\partial V](x)$ converges to critical set

---

Deployment: multi-center optimization

sphere packing and disk covering

“move away from closest”:

$$
\dot{p}_i = + \ln[\partial \text{sm}_Q(p)](p_i)
$$

“move toward furthest”:

$$
\dot{p}_i = - \ln[\partial \text{lg}_Q(p)](p_i)
$$

Aggregate objective functions!

$$
\mathcal{H}_{sp}(P) = \min_i \text{sm}_{V_i}(P)(p_i) = \min_{i \neq j} \left[ \frac{1}{2} ||p_i - p_j||, \text{dist}(p_i, \partial Q) \right]
$$

$$
\mathcal{H}_{dc}(P) = \max_i \text{lg}_{V_i}(P)(p_i) = \max_{q \in Q} \left[ \min_i ||q - p_i|| \right]
$$

---

Deployment: multi-center optimization

Critical points of $\mathcal{H}_{sp}$ and $\mathcal{H}_{dc}$ (locally Lipschitz)

- If $0 \in \text{int}(\partial \mathcal{H}_{sp}(P))$, then $P$ is strict local maximum, all agents have same cost, and $P$ is incenter Voronoi configuration
- If $0 \in \text{int}(\partial \mathcal{H}_{dc}(P))$, then $P$ is strict local minimum, all agents have same cost, and $P$ is circumcenter Voronoi configuration

Aggregate functions monotonically optimized along evolution

$$
\min \tilde{L}_{\ln[\partial \text{sm}_{V_i}]} \mathcal{H}_{sp}(P) \geq 0 \quad \max \tilde{L}_{-\ln[\partial \text{lg}_{V_i}]} \mathcal{H}_{dc}(P) \leq 0
$$

Asymptotic convergence to center Voronoi configurations via nonsmooth LaSalle
Voronoi-circumcenter algorithm

Robotic Network: $S_D$ in $Q$ with absolute sensing of own position

Distributed Algorithm: Vrn-crcmcntr

Alphabet: $A = \mathbb{R}^d \cup \{\text{null}\}$

function $msg(p, i)$
1: return $p$

function $ctl(p, y)$
1: $V := Q \cap \bigcap \{H_{p, prcvd} | \text{for all non-null } p_{rcvd} \in y\}$
2: return $CC(V) - p$

Voronoi-incenter algorithm

Robotic Network: $S_D$ in $Q$, with absolute sensing of own position

Distributed Algorithm: Vrn-ncntr

Alphabet: $A = \mathbb{R}^d \cup \{\text{null}\}$

function $msg(p, i)$
1: return $p$

function $ctl(p, y)$
1: $V := Q \cap \bigcap \{H_{p, prcvd} | \text{for all non-null } p_{rcvd} \in y\}$
2: return $x \in IC(V) - p$

Correctness of the geometric-center algorithms

For $\varepsilon \in \mathbb{R}_>0$, the $\varepsilon$-disk-covering deployment task

$T_{\varepsilon-dc-dply}(P) = \begin{cases} 
\text{true,} & \text{if } \|p[i] - CC(V[i](P))\|_2 \leq \varepsilon, i \in \{1, \ldots, n\}, \\
\text{false,} & \text{otherwise},
\end{cases}$

For $\varepsilon \in \mathbb{R}_>0$, the $\varepsilon$-sphere-packing deployment task

$T_{\varepsilon-sp-dply}(P) = \begin{cases} 
\text{true,} & \text{if } \text{dist}_2(p[i],IC(V[i](P))) \leq \varepsilon, i \in \{1, \ldots, n\}, \\
\text{false,} & \text{otherwise},
\end{cases}$

Theorem

For $d \in \mathbb{N}$, $r \in \mathbb{R}_>0$ and $\varepsilon \in \mathbb{R}_>0$, the following statements hold.

1. on the network $S_D$, any execution of the law $CC_{Vrn-crcmcntr}$ monotonically optimizes the multicenter function $H_{dc}$;
2. on the network $S_D$, any execution of the law $CC_{Vrn-ncntr}$ monotonically optimizes the multicenter function $H_{sp}$.

Summary and conclusions

Aggregate objective functions

1. variety of scenarios: expected-value, disk-covering, sphere-packing
2. smoothness properties and gradient information
3. geometric-center control and communication laws

Technical tools

1. Geometric optimization
2. Geometric models, proximity graphs, spatially-distributed maps
3. Systems theory, nonsmooth stability analysis
References

Deployment scenarios and algorithms:

Nonsmooth stability analysis:

Geometric and combinatorial optimization:

Voronoi partitions

Let \((p_1, \ldots, p_n) \in Q^n\) denote the positions of \(n\) points

The Voronoi partition \(\mathcal{V}(P) = \{V_1, \ldots, V_n\}\) generated by \((p_1, \ldots, p_n)\)

\[
V_i = \{q \in Q \mid \|q - p_i\| \leq \|q - p_j\|, \forall j \neq i\} = Q \cap \mathcal{H}(p_i, p_j)
\]

where \(\mathcal{H}(p_i, p_j)\) is half plane \((p_i, p_j)\)

Distributed Voronoi computation

Assume: agent with sensing/communication radius \(R_i\)

Objective: smallest \(R_i\) which provides sufficient information for \(V_i\)

For all \(i\), agent \(i\) performs:

1: initialize \(R_i\) and compute \(\hat{V}_i = \bigcap_{j} \{p_i - p_j\} \leq R_i \mathcal{H}(p_i, p_j)\)
2: while \(R_i < 2 \max_{q \in \hat{V}_i} \|q - p_i\|\) do
3: \(R_i := 2R_i\)
4: detect vehicles \(p_j\) within radius \(R_i\), recompute \(\hat{V}_i\)